Boundary value problems for elliptic operators with singular drift terms

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

Let Ω be a Lipschitz domain in $\mathbb{R}^n, n \geq 3$, and $L = \mathrm{div} A\nabla - B\nabla$ be a second order elliptic operator in divergence form with real coefficients such that A is a bounded elliptic matrix and the vector field $B \in L^\infty_{loc}(\Omega)$ is divergence free and satisfies the growth condition $\mathrm{dist}(X, \partial\Omega)|B(X)| \leq \varepsilon_1$ for ε_1 small in a neighbourhood of $\partial\Omega$. For these elliptic operators we will study on the basis of the theory for elliptic operators without drift terms the Dirichlet problem for boundary data in $L^p(\partial\Omega), 1 < p < \infty$, and the regularity problem for boundary data in $W^{1,p}(\partial\Omega)$ and HS^1.
The main result of this thesis is that the solvability of the regularity problem for boundary data in HS^1 implies the solvability of the adjoint Dirichlet problem for boundary data in $L^{p'}(\partial\Omega)$ and the solvability of the regularity problem with boundary data in $W^{1,p}(\partial\Omega)$ for some $1 < p < \infty$. In [KP93] C.E. Kenig and J. Pipher have proven for elliptic operators without drift terms that the solvability of the regularity problem with boundary data in $W^{1,p}(\partial\Omega)$ implies the solvability with boundary data in HS^1. Thus the result of C.E. Kenig and J. Pipher and our main result complement a result in [DKP10], where it was shown for elliptic operators without drift terms that the Dirichlet problem with boundary data in BMO is solvable if and only if it is solvable for boundary data in $L^p(\partial\Omega)$ for some $1 < p < \infty$.
In order to prove the main result we will prove for the elliptic operators L the existence of a Green's function, the doubling property of the elliptic measure and a comparison principle for weak solutions, which are well known results for elliptic operators without drift terms.
Moreover, the solvability of the continuous Dirichlet problem will be established for elliptic operators $L = \mathrm{div}(A\nabla + B) + C\nabla + D$ with $B, C, D \in L^\infty_{loc}(\Omega)$ such that in a small neighbourhood of $\partial\Omega$ we have that $\mathrm{dist}(X, \partial\Omega)(|B(X)| + |C(X)| + |D(X)|) \leq \varepsilon_1$ for ε_1 small and that the vector field B satisfies $|\int B\nabla\phi| \leq C\int|\nabla\phi|$ for all $\phi \in W^{1,1}_0$ of that neighbourhood.

Contents

Chapter 1

Introduction

In this thesis, we will study boundary value problems for elliptic operators L with real coefficients in divergence form with singular drift terms and the corresponding real-valued scalar solutions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 3$.
For elliptic operators L_0 of the form $L_0 = \mathrm{div} A\nabla$, where the matrix $A = (a_{ij}(X))$ has real, bounded measurable coefficients such that there exists $\lambda > 0$ with $\lambda|\xi|^2 \leq \sum_{ij} a_{ij}(X)\xi_i\xi_j$ for all $\xi \in \mathbb{R}^n$ and almost every $X \in \Omega$, the Lax-Milgram Theorem implies that for every $f \in W^{\frac{1}{2},2}(\partial\Omega)$ there exists a unique weak solution $u \in W^{1,2}(\Omega)$ with boundary data f, i.e.

$$\int_\Omega A\nabla u \cdot \nabla\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

and $\mathrm{Tr}(u) \equiv f$ on $\partial\Omega$, where Tr is the trace operator. This means that the Dirichlet problem

$$\begin{aligned} L_0 u &= 0 \text{ in } \Omega \\ u &\equiv f \text{ on } \partial\Omega \end{aligned}$$

is solvable for boundary data in $W^{\frac{1}{2},2}(\partial\Omega)$, where the last equality is to be understood in the trace sense. The question if solvability still holds for other classes of boundary values was extensively studied. In [LSW63], it was shown that the continuous Dirichlet problem is solvable for elliptic operators L_0, i.e. for every $f \in C^0(\partial\Omega)$ there exists a unique $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C^0(\bar{\Omega})$ such that $L_0 u = 0$ in Ω and $u \equiv f$ on $\partial\Omega$.
What about boundary data in $L^p(\partial\Omega)$? Historically, the study of the Dirichlet problem with boundary data in $L^p(\partial\Omega)$ for elliptic operators of L_0-type was initiated by B.E.J. Dahlberg in [Dah77], where the Laplacian on Lipschitz domains was considered (the pullback of the Laplacian on a Lipschitz domain leads to an elliptic operator of the form L_0).
As one can see for example in [Dah79], the study of Dirichlet boundary data in $L^p(\partial\Omega)$ is related to the study of the non-tangential maximal function. Due to the fact that the continuous Dirichlet problem is solvable for L_0-type elliptic operators, one defines that the Dirichlet problem with boundary data in $L^p(\partial\Omega)$ is solvable for L_0 (abbreviated $(D)_p$), if for every $f \in C^0(\partial\Omega)$ the weak solution u to the problem $L_0 u = 0$ in Ω and $u \equiv f$ on $\partial\Omega$ satisfies $||u^*||_{L^p(\partial\Omega)} \leq C||f||_{L^p(\partial\Omega)}$, where $(\cdot)^*$ denotes the non-tangential maximal function (see Definitions 5.1.1 and 5.1.2). This $(D)_p$ condition allows one to conclude that for every $f \in L^p(\partial\Omega)$ there exists a unique $u \in W^{1,2}_{loc}(\Omega)$ such that $L_0 u = 0$ and u converges non-tangentially almost everywhere to f on $\partial\Omega$. Thus the question of interest is for which classes of elliptic operators L_0 the $(D)_p$-condition holds.
Apart from the Dirichlet boundary value problem with data in L^p of great interests are also other boundary value problems in particular the L^p Neumann problem and Dirichlet regularity problem (or just regularity problem) where the data are in

$$W^{1,p}(\partial\Omega) = \{f \in L^p(\partial\Omega);\ \nabla_T f \in L^p(\partial\Omega)\},$$

where the unit vectors tangential to the boundary of Ω at Q are $\vec{T}_i(Q)$, or sometimes $\vec{T}(Q)$ to denote the family of these, and $(\int_{\partial\Omega} |\nabla_T f|^p)^{\frac{1}{p}} = (\sum_i \int_{\partial\Omega} |\nabla f(Q) \cdot \vec{T}_i(Q)|^p \, \mathrm{d}\sigma(Q))^{\frac{1}{p}}$.
The most classical method for solving these types of boundary value problems (at least for symmetric operators with coefficients of sufficient smoothness) is the method of layer potentials [FJR78] for the Laplacian in $\mathbb{R}^n$ and [MT99], [MT01], [MT00] for variable coefficients operators. What has been observed are intriguing relationships between various boundary value problems. Of particular note is the duality between the L^p Dirichlet boundary value problem and $W^{1,p'}$ regularity problem (p' denotes the conjugate exponent of p in the whole thesis, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). It turns out that the L^p Dirichlet boundary value problem is solvable if and only if the $W^{1,p'}$ regularity problem is solvable for the same operator (assuming symmetry and sufficient smoothness of the coefficients).
If one does not assume any restrictions on the smoothness of the coefficients nor the symmetry, one has to follow a different path bypassing the shortfalls of the layer potential methods. This path uses some new methods (see for example [KKPT00], Theorem 2.3), certain fundamental properties of weak solutions of elliptic partial differential equations (e.g. the maximum principle, the Harnack inequality) and very sophisticated way of integration by parts. For example in [KKPT00], where the study of non-symmetric divergence form operators was initiated, it is shown in two dimensions that the Dirichlet problem for boundary data in L^p for some (possibly large) $1 < p < \infty$ is solvable if the matrix A is independent in one of the variables.
The results in the literature on the Dirichlet problem for boundary data in $L^p(\partial\Omega)$ can be categorized into three different types (see Chapter 5 for examples and the related papers):

- the solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$ for a certain class of operators is proven directly.
- perturbation results: under the assumption of the $(D)_p$ condition for one specific elliptic operator, the solvability of the Dirichlet problem with boundary data in $L^q(\partial\Omega)$ is proven for a class of elliptic operators which are perturbations of that specific elliptic operator. The index q might be equal to p or larger.
- consequences of the $(D)_p$ condition are proven, e.g. an interpolation and extrapolation property, i.e. that $(D)_p$ implies $(D)_q$ for $q \in (p - \varepsilon, \infty)$ and some $\varepsilon > 0$, which means that solvability is an open property with respect to the index p on $(1, \infty)$.

In contrast to the Dirichlet problem the regularity problem imposes some regularity on the boundary data, e.g. $f \in W^{1,p}(\partial\Omega)$ or $f \in \mathrm{HS}^1(\partial\Omega)$. The study of the regularity problem for elliptic operators $L_0 = \mathrm{div} A\nabla$ with A as above and symmetric was started by C.E. Kenig and J. Pipher in [KP93]. They say that the regularity problem for boundary data in $W^{1,p}(\partial\Omega)$ is solvable (abbreviated by $(R)_p$) if for every $f \in W^{1,p}(\partial\Omega) \cap C^0(\partial\Omega)$ the weak solution u to $L_0 u = 0$ in Ω and $u \equiv f$ on $\partial\Omega$ satisfies $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} \leq C||f||_{W^{1,p}(\partial\Omega)}$, where $N(\cdot)$ is a variant of the non-tangential maximal function. As for the Dirichlet problem, the $(R)_p$ condition allows to conclude that for every $f \in W^{1,p}(\partial\Omega)$ there exists a unique $u \in W^{1,2}_{loc}(\Omega)$ such that $L_0 u = 0$, u converges non-tangentially to f almost everywhere and $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} \leq C||f||_{W^{1,p}(a\partial\Omega)}$. The results on the regularity problem can be categorized into the same three categories as for the Dirichlet problem.
In this thesis we are mostly dealing with two things: first, we will extend several well known results for elliptic operators without drift terms or with drift terms in L^∞ to singular drift terms and, second, we will study consequences of the $(R)_{\mathrm{HS}^1}$ condition, which we will define in Definition 6.2.1.
In [GT01] it is shown that the continuous Dirichlet problem for elliptic operators of the form

$$Lu = \mathrm{div}(A\nabla u + Bu) + C\nabla u + Du$$

is solvable for bounded B, C and D. In [HL01] S. Hofmann and J.L. Lewis consider singular drift terms and it is shown there that the continuous Dirichlet problem is solvable for elliptic operators of the above form with $B, D \equiv 0$ and the vector field C satisfies the growth condition $\mathrm{dist}(X, \partial\Omega)|C(X)| \leq C$ for a constant $C > 0$ and $\mathrm{dist}(X, \partial\Omega)|C(X)|^2 \, \mathrm{d}X$ is a Carleson measure. By applying a scaling argument (see the proof of Theorem 2.2.1) to the results

in [GT01] and an integral version of Hardy's inequality (see Lemma 2.1.2) we are able to prove the solvability of the continuous Dirichlet problem for a similar, but different class of coefficients than in [HL01]. We show that the continuous Dirichlet problem is solvable for singular drift terms of the form $B, C, D \in L^{\infty}_{loc}(\Omega)$ such that in a neighbourhood of $\partial\Omega$ one has $\operatorname{dist}(X, \partial\Omega)(|B(X)| + |C(X)| + |D(X)|) \leq \varepsilon_1$ for ε_1 small and $|\int B\nabla\phi| \leq C\int |\nabla\phi|$ for all $\phi \in W_0^{1,1}$ of that neighbourhood. The family of these elliptic operators is denoted by $\mathcal{O}$. Thus compared to [HL01] we need the smallness of ε_1, but not the Carleson measure condition.
For elliptic operators without drift terms and no assumption on the smoothness of the matrix A properties like the existence of a Green's function, see [GW82], and the doubling property of the elliptic measure, see [CFMS81], are well known and these properties are essential in the study of the theory of boundary values in L^p for elliptic operators without drift terms and no assumption on the smoothness of the matrix A. We will adapt the proofs for these well known properties to the subset $\mathcal{O}_0 \subset \mathcal{O}$, where $L \in \mathcal{O}$ is in $\mathcal{O}_0$ if $C, D \equiv 0$ and B is divergence free. The major difficulty, why we have to restrict ourselves to this subset, is to prove the existence of a Green's function for $L \in \mathcal{O}$ and the adjoint L^* of L. The assumption that $C \equiv 0 \equiv D$ and B is divergence free will make this possible.
The main motivation for this thesis is the paper [DKP10] by M. Dindoš, C.E. Kenig and J. Pipher. They define the $(D)_{BMO}$ condition (for elliptic operators without drift terms), which is the endpoint at ∞ for the $(D)_p$ condition, and they prove that the $(D)_p$ condition is an open condition with respect to the index p on $(1, \infty]$, where $(D)_\infty$ is to be understood as $(D)_{BMO}$. Precisely, they show that $(D)_{BMO}$ holds if and only if the corresponding elliptic measure is in A_∞ and therefore if and only if $(D)_p$ holds for some $1 < p < \infty$.
Motivated by the duality of BMO and H^1, we define for $L \in \mathcal{O}_0$ the solvability of the regularity problem for boundary data in $\mathrm{HS}^1(\partial\Omega)$, abbreviated by $(R)_{\mathrm{HS}^1}$, which can be seen as the natural extension of the definition of the $(R)_p$ condition for $1 < p < \infty$ given in [KP93]. With Theorem 0.1 in [BB10] and the methods in [KP93] we will show that under the assumption of the $(R)_{\mathrm{HS}^1}$ condition for every $f \in \mathrm{HS}^1$ exists a unique weak solution $u \in W^{1,2}_{loc}(\Omega)$ with boundary data f. Moreover, in [KP93] it is shown that $(R)_p$ for $1 < p < \infty$ implies $(R)_{\mathrm{HS}^1}$ (this result is contained in the proof of Theorem 5.2 of [KP93]) and $(R)_{p+\varepsilon}$ for some $\varepsilon > 0$ for symmetric elliptic operators without drift terms. By the characterisation of HS^1 in terms of a maximal function in [BD09] and the methods used in [KP93] and [She07] we will be able to proof the extrapolation property of the $(R)_p$ condition at the endpoint $(R)_{\mathrm{HS}^1}$. Namely, we show that $(R)_{\mathrm{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$. Thus the result of C.E. Kenig and J. Pipher and our extrapolation result, which is the main result of this thesis, complement the result in [DKP10].
The second motivation for investigating the $(R)_{\mathrm{HS}^1}$ condition is the above mentioned duality of the Dirichlet problem with boundary data in $L^{p'}(\partial\Omega)$ and the regularity problem with boundary data in $W^{1,p}(\partial\Omega)$. Since we do not assume symmetry of the matrix A, the question precisely is, if the $(D^*)_{p'}$ condition, which is the $(D)_{p'}$ condition for the adjoint elliptic operator, is equivalent to the $(R)_p$ condition (the direction that $(R)_p$ implies $(D^*)_{p'}$ is proven in [KP93]). The fact that most of the proven results for the Dirichlet problem have been proven for the regularity problem as well (compare for example [Dah79] and [KP93] or [KKPT00] and [KR09]) emphasizes the interest on that open problem. With the aid of Z. Shen's main result in [She07], we are able to simplify the requirements for a possible proof of that duality (see Corollary 6.3.13).

Outline In *Chapter* 2, we will follow the ideas and proofs in [GT01], Chapter 8, and will combine them with a variant of Hardy's inequality to show that the continuous Dirichlet problem on a Lipschitz domain Ω for elliptic operators $L \in \mathcal{O}$ is solvable. Further, we will follow [CFMS81] and [HL01] to extend some results regarding the behaviour of weak solutions for $L \in \mathcal{O}$ at the boundary. In the last section of Chapter 2 we will prove an approximation argument for elliptic operators in $\mathcal{O}$ under the additional assumption that constant functions are weak solutions. This approximation argument originates in [KP93] for $L_0 = \operatorname{div} A\nabla$ and A symmetric and is extended to possibly non-symmetric A in [KKPT00].
In *Chapter* 3, we will prove the existence of a Green's function for elliptic operators in $\mathcal{O}_0$ by

adjusting the corresponding proof in [GW82], which deals with operators without drift terms.
In *Chapter* 4, we will introduce the harmonic measure and, as in [CFMS81], we will prove the doubling property for elliptic measures and a comparison Theorem for weak solutions corresponding to elliptic operators in $\mathcal{O}_0$, which are well-known results for elliptic operators without drift terms.
In *Chapter* 5, we will look at the Dirichlet problem for boundary data in $L^p(\partial\Omega)$ and we will introduce the A_p-weights of Muckenhoupt. We will give a detailed proof based on Young's inequality for Orlicz spaces of a $\mathrm{L}\log\mathrm{L}$–characterization for A_∞ (the endpoint of Gehring's Lemma).
In *Chapter* 6, we will introduce the regularity problem on Lipschitz domains for boundary data in the Hardy–Sobolev space HS^1. We will show that, under the assumption that $(R)_{\mathrm{HS}^1}$ holds, for every $f \in \mathrm{HS}^1$ exists a unique weak solution u such that u converges non-tangentially to f almost everywhere. Further, we will show that $(R)_{\mathrm{HS}^1}$ implies $(D^*)_{p'}$ and $(R)_p$ for some $1 < p < \infty$. In addition, we will introduce the $(R)_{\mathcal{C}^q}$ condition for $q < 1$ and look at the extrapolation property of the Neumann problem on the Hardy space H^1 (the extrapolation property for $(N)_p$ with $1 < p < \infty$ is proven in [KP93]).
In the last chapter, *Chapter* 7, we summarize some open problems, which appear in this thesis and make suggestions for some further investigations.

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Chapter 2

Elliptic Operators in Divergence Form

In this chapter we consider elliptic operators in divergence form with singular drift terms on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 3$, i.e. operators of the form

$$Lu(X) = \operatorname{div}(A(X)\nabla u(X) + B(X)u(X)) + C(X)\nabla u(X) + D(X)u(X) \tag{2.1}$$

where A is a real bounded elliptic matrix and real $B, C, D \in L^\infty_{loc}(\Omega)$ such that in a neighbourhood of $\partial\Omega$ one has $\operatorname{dist}(X, \partial\Omega)(|B(X)| + |C(X)| + |D(X)|) \leq \varepsilon_1$ for ε_1 small and $|\int B\nabla\phi| \leq C\int |\nabla\phi|$ for all $\phi \in W^{1,1}_0$ in that neighbourhood (see the beginning of section 2.2 for the precise definition).
A function $u \in W^{1,2}_{loc}(\Omega)$ (the function space $W^{1,2}_{loc}(\Omega)$ and the domain Ω are defined in section 2.1) is called a weak solution (subsolution, supersolution) for L if $Lu = 0(\geq 0, \leq 0)$ in the weak sense, i.e.

$$\mathcal{L}(u, v) = \int A\nabla u \cdot \nabla v + u\, B\nabla v - C\nabla u\, v - Duv = 0(\leq 0, \geq 0) \tag{2.2}$$

for all $v \in C^1_0(\Omega)$ with $v \geq 0$.
Using a scaling argument we see that a weak solution u for $L \in \mathcal{O}$ can locally on a ball centred at $X \in \Omega$ with radius $\alpha = \operatorname{dist}(X, \partial\Omega)/2$ be seen like a weak solution u_α for L_α on a ball of radius one which is approximately one away from the boundary, where L_α has bounded coefficients $B_\alpha, C_\alpha, D_\alpha$ on that ball (see Theorem 2.2.1 for the precise argument). This means that we can use the local results from [GT01], Chapter 8, where operators as in $\mathcal{O}$ but with bounded B, C, D are considered. One can see section 2.2 as a generalization of Chapter 8 in [GT01], especially since we will adapt the proofs from [GT01] with the aid of Lemma 2.1.2 to show that the continuous Dirichlet problem is solvable for our type of elliptic operators.
For elliptic operators with bounded drift terms it is shown in [GT01] and for example in [Ken94] (where the proof is given for operators of the form $\operatorname{div}A\nabla$, but can easily be extended to operators of the form (2.1) with bounded drift terms) that weak solutions in the interior of Ω are Hölder continuous, satisfy the Harnack principle and the Cacciopoli inequality. Moreover, it is shown that if $\int_\Omega (Dv - B\nabla v) \leq 0$ or $\int_\Omega (Dv + C\nabla v) \leq 0$ for all non-negative $v \in C^1_0(\Omega)$, the maximum principle holds, and therefore that under the additional assumption that Ω satisfies an exterior cone condition at every $Q \in \partial\Omega$, one can find a unique $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C^0(\bar{\Omega})$ for every $g \in C^0(\partial\Omega)$ with

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \\ u &\equiv g \text{ on } \partial\Omega. \end{aligned}$$

This means that the continuous Dirichlet problem is solvable.
In [HL01] S. Hofmann and J.L. Lewis consider operators of the form $Lu = \operatorname{div}A\nabla u + C\nabla u$ for A as above and the vector field C satisfies $\operatorname{dist}(X, \partial\Omega)|C(X)| \leq c$ for some $c > 0$ and

$\mathrm{dist}(X, \partial\Omega)|C(X)|^2 \, \mathrm{d}X$ is a Carleson measure. Among other things they show that the continuous Dirichlet problem is solvable for these operators (see [KP01] as well). Thus if one compares the operators in [HL01] and our operators in $\mathcal{O}$ one sees that we require the smallness of ε_1, but we do not impose the Carleson measure condition.
The main result of this chapter is that the continuous Dirichlet problem is solvable for operators in $\mathcal{O}$. We will start this chapter by defining the spaces $W^{k,p}(\Omega)$ and the domain Ω. Then we show the solvability of the continuous Dirichlet problem for elliptic operators in $\mathcal{O}$. Furthermore, we will give detailed proof for some results about the behaviour of weak solutions at the boundary, which are well known for operators without drift terms. At the end of this chapter we prove an approximation argument for elliptic operators in a subclass of $\mathcal{O}$. This approximation argument originates in [KP93], section 7, for elliptic operators of the form $L_0 = \mathrm{div} A \nabla$ and A symmetric and is extended to non-symmetric A in [KKPT00], page 257. The proof given in [KKPT00] relies on the coercivity of L_0. We will bypass the lack of coercivity by the usage of the Green's operator.

2.1 The spaces $W^{k,p}$ and Lipschitz Domains

In this section, we summarize well-known facts about the function spaces $W^{k,p}$, Lipschitz domains and the trace operator.
Let $\Omega \subset \mathbb{R}^n$ be bounded and open. The dimension n will be larger than or equal to 3 in the whole thesis, if it is not stated otherwise. For $u \in L^1_{loc}(\Omega)$, we say that v is the α^{th}-weak derivative of u (with $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index) if

$$\int_\Omega v\varphi = (-1)^{|\alpha|} \int_\Omega u \varphi^{(\alpha)}$$

for all $\varphi \in C_0^{|\alpha|}(\Omega)$ where $|\alpha| = \sum_{j=1}^n \alpha_j$. We denote the weak derivative v of u by $D^\alpha u$. Therefore $D^\alpha u$ is defined almost everywhere. A function is called weakly differentiable of order k if all α^{th}-weak derivatives exist for all $|\alpha| \leq k$. An integration by parts argument shows that, if $u \in C^k(\Omega)$, the weak derivative of u coincides with the derivative of u for all α with $|\alpha| \leq k$. The subspace of $L^1_{loc}(\Omega)$ of all weakly differentiable functions of order k for k an integer is denoted by $W^k(\Omega)$ and we define the space $W^{k,p}(\Omega)$, $1 \leq p < \infty$, by

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}$$

and a norm on $W^{k,p}(\Omega)$ by

$$||u||_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p(\Omega)}.$$

The completion of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$. We will use the following notation throughout the thesis: Q, P denote points on $\partial\Omega$ and X, Y in Ω. For $Z \in \mathbb{R}^n$ and $R > 0$, the open ball centred at Z with radius R is denoted by $B_R(Z)$ and $r(B)$ denotes the radius of the ball B. In addition

$$\begin{aligned}
\Delta_R(Q) &= \partial\Omega \cap B_R(Q), \\
T_R(Q) &= \Omega \cap B_R(Q), \\
\delta(X) &= \mathrm{dist}(X, \partial\Omega), \\
(\partial\Omega)_\beta &= \{X \in \Omega : \delta(X) < \beta\}, \\
\Omega_\beta &= \Omega \backslash (\partial\Omega)_\beta.
\end{aligned}$$

For $f \in L^1(E)$ with E a measurable set with positive Lebesgue measure, i.e. $|E| > 0$, we write $f(E) = \int_E f$ and $f_E = \fint_E f = \frac{1}{|E|} \int_E f$.

Definition 2.1.1 ($C^{k,\alpha}$-Domain; [GT01], page 94)**.** *We call a bounded domain $\Omega \in \mathbb{R}^n$ a $C^{k,\alpha}$-domain, if at each point $Q \in \partial\Omega$ there exists a ball $B_{r(Q)}(Q)$ and a one-to-one mapping ψ_Q*

onto the unit ball $B_1(0) \subset \mathbb{R}^n$ such that:

- $\psi_Q(B_{r(Q)}(Q) \cap \Omega) = B_1(0) \cap \mathbb{R}^n_+$
- $\psi_Q(B_{r(Q)}(Q) \cap \partial\Omega) = B_1(0) \cap \partial\mathbb{R}^n_+$
- $\psi_Q \in C^{k,\alpha}(B_{r(Q)}(Q))$, $\psi_Q^{-1} \in C^{k,\alpha}(B_1(0))$,

where $\mathbb{R}^n_+ = \{(x', t) : x' \in \mathbb{R}^{n-1}, t > 0\}$.

Lemma 2.1.1. *Let Ω be a $C^{0,1}$-domain. Then there exists a finite sequence $\{P_j\}_j \in \partial\Omega$ and a constant $R_0 > 0$ such that for every $Q \in \partial\Omega$ there exists a P_j with $T_{R_0}(Q) \subset T_{r(P_j)}(P_j)$, where $r(P_j)$ is given by Definition 2.1.1. Hence there exists a finite sequence $\{Q_k\}_k \in \partial\Omega$ such that $(\partial\Omega)_{R_0/2} \subset \bigcup_k T_{R_0}(Q_k)$.*

Proof. Since $\partial\Omega$ is compact we can find a finite sequence $\{P_j\} \in \partial\Omega$ such that

$$\bigcup_j \psi_{P_j}^{-1}(\partial\mathbb{R}^n_+ \cap B_{\frac{1}{2}}(0)) = \partial\Omega.$$

All $\psi_{P_j}, \psi_{P_j}^{-1}$ are Lipschitz continuous and therefore, we can find a uniform constant M such that

$$\frac{1}{M}|X - Y| \leq |\psi_{P_j}^{-1}(X) - \psi_{P_j}^{-1}(Y)| \leq M|X - Y|$$

for all $X, Y \in B_1(0)$. This means that the image of a ball $B_r(Z) \subset B_1(0)$ under any $\psi_{P_j}^{-1}$ contains a ball with radius at least $\frac{1}{M}r$. Thus for $Z \in \partial\mathbb{R}^n_+ \cap B_{\frac{1}{2}}(0)$, the set $\psi_{P_j}^{-1}(B_{\frac{1}{2}}(Z))$ contains the ball $B_{\frac{1}{2M}}(\psi_{P_j}^{-1}(Z))$. Hence, for every $Q \in \partial\Omega$ there exists j such that $B_{\frac{1}{2M}}(Q) \subset \psi_{P_j}^{-1}(B_1(0))$, which implies that $T_{\frac{1}{2M}}(Q) \subset T_{r(P_j)}(\psi_{P_j}^{-1}(0))$. The choice

$$R_0 = \frac{1}{2M} \tag{2.3}$$

finishes the first part of the proof.
We have $\overline{(\partial\Omega)}_{R_0/2} \subset \bigcup_{Q \in \partial\Omega} B_{R_0}(Q)$. A compactness argument justifies the existence of a finite sequence $\{Q_k\}_k \in \partial\Omega$ such that $(\partial\Omega)_{R_0/2} \subset \bigcup_k T_{R_0}(Q_k)$, which completes the proof. $\square$

Definition 2.1.2. *A $C^{0,1}$-domain is called a Lipschitz domain.*

For Ω a Lipschitz domain, we see that Ω is locally the region above a Lipschitz graph φ and so for $Q = (x', \varphi(x')) \in \partial\Omega$ we define $A_R(Q) = (x', \varphi(x') + R)$ and for $X \in \Omega$ we define $\hat{X} \in \partial\Omega$ such that $A_R(\hat{X}) = X$ for an appropriate R. Thus $A_R(Q)$ and $\hat{X}$ are well defined in each $\Omega \cap B_{R_0}(Q_k)$, where R_0 and Q_k are as in Lemma 2.1.1. This means that $A_R(Q)$ and $\hat{X}$ depend on k, but we will omit the index k to maintain an easy readable notation.
The next lemma follows from the Hardy inequality, which originates in [Har20], (4). An integral version of Hardy's inequality says that if f is a non-negative integrable function then $\int_0^\infty (\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s)^p\,\mathrm{d}t \leq (\frac{p}{p-1})^p \int_0^\infty f(t)^p\,\mathrm{d}t$ for $p > 1$.

Lemma 2.1.2. *Let Ω be a Lipschitz domain and B be a non-negative measurable function in Ω with $B(X) \leq \frac{\varepsilon_1}{\delta(X)}$ for some $\varepsilon_1 > 0$ in $(\partial\Omega)_\beta$ and $B(X) \leq C$ in Ω_β. Then for $\varphi \in W_0^{1,s}(\Omega)$ non-negative, $v \in W^{1,s'}(\Omega)$, $1 < s < \infty$, and any $R < \min\{\beta, R_0\}$, $Q_0 \in \partial\Omega$ we have*

- $\int_{T_R(Q_0)} (B\varphi)^s \leq C_s \varepsilon_1^s \int_{T_R(Q_0)} |\nabla\varphi|^s$
- $\int_\Omega |\nabla v| B\varphi\phi^2 \leq \varepsilon \int_\Omega |\nabla v|^{s'} \phi^{s'} + C_{s,\varepsilon}\varepsilon_1^s \int_{(\partial\Omega)_\beta} (|\nabla\varphi|^s \phi^s + \varphi^s |\nabla\phi|^s) + C_{s,\varepsilon} \int_{\Omega_\beta} |\varphi\phi|^s$

where ϕ is any non-negative, smooth function in $\mathbb{R}^n$ and $\varepsilon > 0$.

Proof. The first inequality follows from the integral version of the Hardy inequality and the second from the first and an application of Young's inequality. $\square$

In order to introduce the trace operator, let us observe that

Theorem 2.1.3 ([GT01], Theorem 7.25). *Let Ω be a $C^{k-1,1}$-domain. Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega), 1 \le p < \infty$.*

For $\phi \in C^\infty(\bar{\Omega})$ the map $\mathrm{Tr} : C^\infty(\bar{\Omega}) \to C^0(\partial\Omega)$ defined by $\mathrm{Tr}(\phi)(Q) = \phi(Q)$ for $Q \in \partial\Omega$ is well defined and is called the trace operator. From the results in [Ada75], section VII, the trace operator is bounded from $W^{k,p}(\Omega)$ to $W^{k-\frac{1}{p},p}(\partial\Omega)$ for Ω a $C^{k-1,1}$-domain (for the definition of fractional order Sobolev spaces see [Ada75]). We know from Theorem 2.1.3 that $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ and so one can extend the trace operator to a bounded linear operator, which we call Tr as well, on $W^{k,p}(\Omega)$ with

$$\mathrm{Tr} : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$$
$$||\mathrm{Tr}(u)||_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le C||u||_{W^{1,p}(\Omega)}$$

for $1 < p < \infty$ and Ω a Lipschitz domain. Roughly speaking, this means that by going to the boundary, one looses $\frac{1}{p}$-th derivatives. By Theorem 7.55 in [Ada75] $W_0^{1,p}(\Omega)$, which was defined as the completion of $C_0^\infty(\Omega)$ under the norm of $W^{k,p}(\Omega)$, coincides with $\{\phi \in W^{1,p}(\Omega) : \mathrm{Tr}(\phi) = 0\}$.
In section 2.2, we will need the following results about traces.

Lemma 2.1.4. *Let Ω be a Lipschitz domain, $u \in W_0^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$, then $uv \in W_0^{1,1}(\Omega)$.*

Proof. There exist $u_k \in C_0^\infty(\Omega)$ and $v_k \in C^\infty(\bar{\Omega})$ with $u_k \to u, v_k \to v$ in $W^{1,2}(\Omega)$ and $||u_k||_{W_0^{1,2}(\Omega)} \le C||u||_{W_0^{1,2}(\Omega)}$. Since $|u_k v_k - uv| \le |u_k||v_k - v| + |v||u - u_k|$ we get $u_k v_k \to uv$ in $W^{1,1}(\Omega)$. Additionally, $u_k v_k \in C_0^\infty(\Omega)$ and so $uv \in W_0^{1,1}(\Omega)$. □

Lemma 2.1.5. *Let Ω be a Lipschitz domain and $u \in W^{1,p}(\Omega)$ be non-negative almost everywhere. Assume that $u \le 0$ on $\partial\Omega$ (understood in the trace sense) then $u \in W_0^{1,p}(\Omega)$.*

Proof. By the technique of mollifiers, we can choose non-negative $\phi_k \in C^\infty(\bar{\Omega})$ with $\phi_k \to u$ in $W^{1,p}(\Omega)$. The definition of the trace operator implies $Tr(\phi_k) \ge 0$ and the boundedness of the trace operator gives $||Tr(u - \phi_k)||_{W^{1-\frac{1}{p}}(\partial\Omega)} \to 0$ as $k \to \infty$. Hence $\mathrm{Tr}(u) \ge 0$ and therefore $Tr(u) = 0$. □

Lemma 2.1.6. *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $M = \sup_{\partial\Omega} u$, where the $\sup_{\partial\Omega}$ is understood in the trace sense, then $\max\{u, M\} - M \in W_0^{1,p}(\Omega)$.*

Proof. By Lemma 2.1.5 we have $|M| < \infty$. Moreover, $\max\{u, M\} - M \ge 0$ in Ω. The definition of M implies $\max\{u, M\} - M \le 0$ on $\partial\Omega$ and, by Lemma 2.1.5, $\max\{u, M\} - M \in W_0^{1,p}(\Omega)$. □

2.2 The Continuous Dirichlet Problem

In this section, we will use Lemma 2.1.2 to extend the results in [GT01], Section 8, from bounded to singular drift terms. We deal with elliptic operators in divergence form L as in (2.1) on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ (see Definition 2.1.2), $n \ge 3$, with real, measurable coefficients that satisfy[1]:

- there exists $\lambda > 0$ such that $\langle A(X)\xi, \xi\rangle \ge \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and almost every $X \in \Omega$.
- there exists $\beta > 0, \varepsilon_1 > 0$ and $M > 0$, with ε_1 small such that
  - $\delta(X)(|B(X)| + |C(X)| + |D(X)|) \le \varepsilon_1$ for almost all $X \in (\partial\Omega)_\beta$
  - $||A||_{L^\infty(\Omega)}, ||B||_{L^\infty(\Omega_\beta)}, ||C||_{L^\infty(\Omega_\beta)}, ||D||_{L^\infty(\Omega_\beta)} \le M$
  - the vector field B satisfies $|\int_{(\partial\Omega)_\beta} B\nabla\varphi| \le M||\nabla\varphi||_{L^1((\partial\Omega)_\beta)}$ for all $\varphi \in C_0^1((\partial\Omega)_\beta)$

[1] In order not to confuse readers, which are familiar with the notation used in the literature, we keep the notation which is used in the literature, although this leads to a clash in the notation. For example D can denote a function or the derivative. It will be always clear from the context or will be explained, which meaning is to be considered.

- positive constants are supersolutions, i.e. $\int_\Omega (Dv - B\nabla v) \leq 0$ for all non-negative $v \in C_0^1(\Omega)$ (this implies by the result in [GT01] that a local maximum principle holds)

The family of elliptic operators that satisfy the above criteria is denoted by $\mathcal{O} = \mathcal{O}(\lambda, \beta, \varepsilon_1, M)$. A function $u \in W^{1,2}_{loc}(\Omega)$ is called a weak solution (subsolution, supersolution) for $L \in \mathcal{O}$ if $Lu = 0 (\geq 0, \leq 0)$ in the weak sense, i.e.

$$\mathcal{L}(u,v) = \int A\nabla u \cdot \nabla v + u\, B\nabla v - C\nabla u\, v - Duv = 0 (\leq 0, \geq 0)$$

for all $v \in C_0^1(\Omega)$ with $v \geq 0$.
The subfamily of operators in $\mathcal{O}$ with $C \equiv 0 \equiv D$ and the vector field B being divergence free in the sense of distributions is denoted by $\mathcal{O}_0$. Thus an operator $L \in \mathcal{O}_0$ is of the form

$$Lu = \mathrm{div}(A\nabla u + Bu) = \mathrm{div} A\nabla u - B\nabla u.$$

The symbol $\approx$ is used as an abbreviation for the following: We will write $f \approx g$ for two functions defined on a given set E, in words that f is comparable with g, if there exists a constant C which depends on $\lambda, \beta, \varepsilon_1, M, \Omega$ and n such that $\frac{1}{C} f \leq g \leq Cf$. We use C_ζ or $C(\zeta)$ if we would like to emphasize that the constant C depends on the parameter ζ.
Let us make a few comments on the restrictions imposed on the operators in $\mathcal{O}$: For the proofs in this section it will be essential that for $L \in \mathcal{O}$ the corresponding bilinear form $\mathcal{L} : W^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ is bounded. This is the main reason for the restrictions imposed on the drift terms. We will give brief examples to show that the growth condition of C and D cannot be relaxed to allow a growth like $\frac{1}{\delta(X)^{1+\varepsilon}}$ for some $\varepsilon > 0$. To illustrate the ideas, let us assume that we are on $(0,1) \subset \mathbb{R}$ and that we restrict the view to the region close to 0. We will write $\int_0$ to mean the integral over the interval $(0, c_0)$ for some $c_0 > 0$ small. If we assume that $C(x) = \frac{1}{x^{1+\varepsilon}}$ and we choose $f(x) = x$ and $g(x) = x^\varepsilon$ (where f symbolises the function in $W^{1,2}(\Omega)$ and g the function in $W_0^{1,2}(\Omega)$) then $\int_0 C(x)f'(x)g(x) = \int_0 \frac{1}{x} = \infty$. Similar thoughts work for D if we take $f \equiv 1$ instead.
For the term involving B it is different, since the derivative is applied to the function from $W_0^{1,2}(\Omega)$. Thus if we assume that $B(x) = \frac{1}{x}, f \equiv 1$ and $g(x) = x$, then $\int_0 B(x)f(x)g'(x) = \int_0 \frac{1}{x} = \infty$. This example shows that in order for $\mathcal{L} : W^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ to be bounded we need an additional assumption on B. For $\phi \in C^\infty(\bar{\Omega})$ and $\psi \in C_0^\infty(\Omega)$ we have $\int_\Omega B\nabla\phi\ \psi = \int B\nabla(\phi\psi) - \int \phi\ B\nabla\psi$. The second term is bounded by $C||\phi||_{W^{1,2}(\Omega)}||\psi||_{W_0^{1,2}(\Omega)}$. Therefore for $\mathcal{L}$ to be a bounded bilinear functional on $W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ the vector field B has to satisfy $|\int_{(\partial\Omega)_\beta} B\nabla\varphi| \leq M||\varphi||_{W_0^{1,1}((\partial\Omega)_\beta)}$ for all $\varphi \in W_0^{1,1}((\partial\Omega)_\beta)$ and some $M > 0$. Obviously, bounded vector fields B and vector fields B with bounded divergence (use the Poincare inequality) satisfy this assumption (with a possibly different M). Thus our assumption on B is weaker than B being bounded or having bounded divergence. Let us give an example of an unbounded vector field B with unbounded divergence that satisfies the assumption as well. For this let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square and define $B(x,y) = (x^\alpha, \frac{1}{x})$ for some $0 < \alpha < 1$. Then for any $\phi \in C_0^\infty(\Omega)$ we have $\int_\Omega B\nabla\phi = \int_\Omega x^\alpha \partial_x \phi(x,y)$. Thus $|\int_\Omega B\nabla\phi| \leq \int_\Omega |\nabla\phi|$ for all $\phi \in W_0^{1,1}(\Omega)$, but B and the divergence of B are unbounded. If we choose $\alpha = 0$, then we see that B grows like $\frac{1}{\delta(x)}$ for $x \to 0$ and B has zero divergence.
The goal of this section is to prove that the continuous Dirichlet problem for elliptic operators in $\mathcal{O}$ is solvable for ε_1 small enough. If one looks at the proofs in [GT01] needed to prove the solvability of the continuous Dirichlet problem for operators with bounded drift terms, one realizes that they can be used in combination with Lemma 2.1.2 almost equally well for our class of elliptic operators.
We start with an interior result. From the results for bounded coefficients in [GT01] and for example [Ken94] we deduce the following:

Theorem 2.2.1. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a non-negative weak solution for $L \in \mathcal{O}$ in the Lipschitz domain Ω. Then u is Hölder continuous and it satisfies the Harnack principle and the Cacciopoli inequality in the interior of Ω.*

Proof. The proof follows from a scaling argument. For X in Ω let $\alpha = \frac{\delta(X)}{2}$ and $\tilde{B}_\alpha$ be the ball with radius α and centre X. We write $f_\alpha(X) = f(\alpha X)$. Then for $\phi \in W_0^{1,2}(\Omega)$ and $X = \alpha Z$ we get

$$\begin{aligned} 0 &= \int_\Omega A\nabla u \cdot \nabla\phi + u\, B\nabla\phi - C\nabla u\, \phi - Du\phi \\ &= \int_{\Omega^{\frac{1}{\alpha}}} A_\alpha \nabla u_\alpha \cdot \nabla\phi_\alpha \frac{1}{\alpha} + u_\alpha\, B_\alpha\nabla\phi_\alpha - C_\alpha\nabla u_\alpha\, \phi_\alpha - D_\alpha u_\alpha \phi_\alpha \alpha, \end{aligned}$$

where $\Omega^{\frac{1}{\alpha}} = \{Z \in \mathbb{R}^n : \alpha Z \in \Omega\}$. Hence u_α is a solution to $L_\alpha u = \operatorname{div}(A_\alpha \nabla u + \alpha B_\alpha u) + \alpha C_\alpha \nabla u + \alpha^2 D_\alpha u$ on $\Omega^{\frac{1}{\alpha}}$. The ball $\tilde{B}_\alpha$ is transformed by the change of variables $X = \alpha Z$ to $\tilde{B}_1$, a ball with radius and distance to the boundary comparable to one. Since for example $\alpha|C_\alpha(Z)| = \alpha|C(\alpha Z)| \approx 1$ for all $Z \in \tilde{B}_1$, the coefficients $A_\alpha, \cdots, D_\alpha$ are bounded on $\tilde{B}_1$. The results for bounded coefficients are applicable. □

Theorem 2.2.1 still holds if B, C, D satisfy $\delta(X)^2|D(X)| \leq C$ and $\delta(X)(|B(X)|+|C(X)|) \leq C$ for some (possibly large) $C > 0$. But we will see in the following proofs that the methods we use rely heavily on the smallness of ε_1 and the boundedness of the bilinear functional $\mathcal{L}$ for $L \in \mathcal{O}$ to prove that the continuous Dirichlet problem for $L \in \mathcal{O}$ is solvable.
Following Lemma 3.38 in [HL01], we get:

Theorem 2.2.2 (Maximum Principle)**.** *Let $u, v \in W^{1,2}_{loc}(\Omega)$ be two weak solutions for $L \in \mathcal{O}$ with $\limsup_{X\to Q}(u - v)(X) \leq 0$ in the Lipschitz domain Ω, then $u \leq v$ in Ω.*

Proof. Since the coefficients of L are locally in L^∞, we get $u, v \in C^0(\Omega)$. Using a compactness argument, we get the existence of a $\delta = \delta(\varepsilon) > 0$ for every $\varepsilon > 0$ such that $u - v \leq \varepsilon$ in $(\partial\Omega)_\delta$. Assume that there exists $X \in \Omega$ such that $(u - v)(X) > 2\varepsilon$. Then $\delta(X) > \delta$ and by the local maximum principle we get $\sup_{\partial(\Omega_\delta)}(u - v) > 2\varepsilon$, which is a contradiction. □

Lemma 2.2.3. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}$. Then $\mathcal{L}(\cdot, \cdot)$ is a bounded bilinear functional on $W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, i.e. for any $\varphi \in W^{1,2}(\Omega)$ we have that $L\varphi$ (which we will sometimes denote as F_φ) is in $(W_0^{1,2}(\Omega))^*$ with $||L\varphi||_{(W_0^{1,2}(\Omega))^*} \leq C||\varphi||_{W^{1,2}(\Omega)}$.*

Proof. It is enough to show that $\mathcal{L}(\cdot, \cdot)$ is a bounded bilinear functional on $W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. Let ψ be smooth with $\psi \equiv 1$ in $(\partial\Omega)_{\beta/4}$, $\psi \equiv 0$ in Ω_β and $|\nabla\psi| \leq C_\beta$. Then

$$\begin{aligned} \mathcal{L}(\varphi, v) &= \int_\Omega A\nabla\varphi \cdot \nabla v + B\nabla(\varphi v) - \nabla(B + C)\varphi\, v - D\varphi v \\ &= \int_\Omega A\nabla\varphi \cdot \nabla v + B\nabla(\varphi v\psi) + B\nabla(\varphi v(1 - \psi)) - (B + C)\nabla\varphi\, v - D\varphi v. \end{aligned}$$

Except for the term involving A we split the integral into $\int_\Omega = \int_{(\partial\Omega)_\beta} + \int_{\Omega_\beta}$. On Ω_β, we use the L^∞ bounds of the coefficients. On $(\partial\Omega)_\beta$, we use the assumption on B, the Cauchy–Schwarz inequality, and Lemma 2.1.2 to get

$$\begin{aligned} &|\int_{(\partial\Omega)_\beta} B\nabla(v\varphi\psi)| \leq C||\nabla(\varphi v\psi)||_{L^1((\partial\Omega)_\beta)} \leq C||\varphi||_{W^{1,2}(\Omega)}\, ||v||_{W_0^{1,2}(\Omega)}, \\ &|\int_\Omega (B + C)\nabla\varphi\, v| \leq C||\varphi||_{W^{1,2}(\Omega)}\, ||v||_{W_0^{1,2}(\Omega)}. \end{aligned}$$

Similar thoughts work for the other terms. Thus $|\mathcal{L}(\varphi, v)| \leq C||\varphi||_{W^{1,2}(\Omega)}\, ||v||_{W_0^{1,2}(\Omega)}$ and so $||L\varphi||_{(W_0^{1,2}(\Omega))^*} \leq C||\varphi||_{W^{1,2}(\Omega)}$.

□

Lemma 2.2.4. *Let Ω be a Lipschitz domain. For $L \in \mathcal{O}$ with ε_1 sufficiently small, there exists $\sigma = \sigma(\mathcal{O}) > 0$ such that the corresponding bilinear form $\mathcal{L}$ satisfies $\mathcal{L}(u, u) \geq \frac{\lambda}{2}\int_\Omega |\nabla u|^2 - \sigma\int_\Omega |u|^2$, where λ is the ellipticity constant of the matrix A.*

Proof. Using Lemma 2.1.2, we see that

$$\begin{aligned}\mathcal{L}(u,u) &\geq \lambda \int_\Omega |\nabla u|^2 - \int_\Omega |\nabla u||B-C||u| - |\int_\Omega Du^2| \\ &\geq \lambda \int_\Omega |\nabla u|^2 - \frac{\lambda}{10}\int_\Omega |\nabla u|^2 - C_\lambda \int_{(\partial\Omega)_\beta} |B-C|^2 u^2 - C_\lambda \int_{\Omega_\beta} |B-C|^2 u^2 \\ &\quad - \int_{(\partial\Omega)_\beta} |D|u^2 - \int_{\Omega_\beta} |D|u^2 \\ &\geq \frac{9\lambda}{10}\int_\Omega |\nabla u|^2 - C_\lambda \varepsilon_1 \int_\Omega |\nabla u|^2 - 5M^2 \int_\Omega u^2,\end{aligned}$$

i.e. for ε_1 small enough, we get $\mathcal{L}(u,u) \geq \frac{\lambda}{2}\int_\Omega |\nabla u|^2 - 5M^2 \int_\Omega u^2$. □

The proofs of the following two theorems use the ideas of the proof of Theorem 8.3 in [GT01].

Theorem 2.2.5. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}$. Then for $F \in (W_0^{1,2}(\Omega))^*$, there exists a unique $w \in W_0^{1,2}(\Omega)$ such that $Lw = F$. Additionally, $||w||_{W_0^{1,2}(\Omega)} \leq C||F||_{(W_0^{1,2}(\Omega))^*}$. If $F = F_\varphi$ then $||w||_{W_0^{1,2}(\Omega)} \leq C||\varphi||_{W^{1,2}(\Omega)}$. The bounded linear operator that maps $(W_0^{1,2}(\Omega))^* \ni F \mapsto w \in W_0^{1,2}(\Omega)$ is called Green's operator and will be denoted by $\mathcal{G}$.*

Proof. Lemma 2.2.4 implies that there exists $\sigma = \sigma(\mathcal{O})$ such that the bilinear form $\mathcal{L}_\sigma$ corresponding to $L_\sigma w = Lw - \sigma w$ is bounded and coercive.
Define the embedding $I : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by $Iw(v) = \int_\Omega wv$. Then I is a compact operator. The equation $Lw = F$ for $F \in (W_0^{1,2}(\Omega))^*$ and $w \in W_0^{1,2}(\Omega)$ is equivalent to

$$L_\sigma w + \sigma I w = F.$$

By the Lax-Milgram Theorem the operator L_σ is invertible with bounded inverse

$$L_\sigma^{-1} : (W_0^{1,2}(\Omega))^* \to W_0^{1,2}(\Omega).$$

Applying L_σ^{-1} to both sides from the left, we get the equivalent formulation

$$w + \sigma L_\sigma^{-1} I w = L_\sigma^{-1} F. \tag{2.4}$$

Since I is compact and L_σ^{-1} is continuous, the operator $(-\sigma L_\sigma^{-1} I) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is compact. Moreover, the equation $w - (-\sigma L_\sigma^{-1} I)w = 0$ is equivalent to $Lw = 0$. Thus the maximum principle, Theorem 2.2.2, implies that only the trivial solution satisfies $w - (-\sigma L_\sigma^{-1} I)w = 0$. The Fredholm alternative, see Theorem A.0.24 in the Appendix, implies that there exists a unique function $w \in W_0^{1,2}(\Omega)$ such that (2.4) holds and the operator $(\mathrm{id} - (-\sigma L_\sigma^{-1} I))$ has a bounded inverse. Thus

$$w = (\mathrm{id} - (-\sigma L_\sigma^{-1} I))^{-1} L_\sigma^{-1} F.$$

with $||w||_{W_0^{1,2}(\Omega)} \leq C||F||_{(W_0^{1,2}(\Omega))^*}$. If $F = F_\varphi$ we get by Lemma 2.2.3 that $||w||_{W_0^{1,2}(\Omega)} \leq C||\varphi||_{W^{1,2}(\Omega)}$ and so the proof is complete. □

Theorem 2.2.6. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}$. Then for every $\varphi \in W^{1,2}(\Omega)$, there exists a unique $u \in W^{1,2}(\Omega)$ such that $Lu = 0$ and $u \equiv \varphi$ on $\partial\Omega$. Furthermore, $||u||_{W^{1,2}(\Omega)} \leq C||\varphi||_{W^{1,2}(\Omega)}$.*

Proof. Let $w \in W_0^{1,2}(\Omega)$ be the solution to $Lw = L\varphi$. The existence of w is guaranteed by Lemma 2.2.3 and Theorem 2.2.5. Define $u = \varphi - w$. So $Lu = 0$ in Ω and $u \equiv \varphi$ on $\partial\Omega$. Lemma 2.2.3 implies $||u||_{W^{1,2}(\Omega)} \leq C||\varphi||_{W^{1,2}(\Omega)}$. The maximum principle gives uniqueness. □

An important step for the proof of the solvability of the continuous Dirichlet problem for elliptic operators with bounded drift terms in [GT01] is Theorem 8.25 in [GT01]. We will combine the proof of Theorem 8.25 in [GT01] with Lemma 2.1.2 to get the result which is needed to conclude the solvability of the continuous Dirichlet problem for $L \in \mathcal{O}$. Moreover we include a new result, which follows from the same methods, for indices smaller one, which will be used later to deal with $(R)_q$ for $q < 1$.

Theorem 2.2.7. *Let Ω be a Lipschitz domain, $L \in \mathcal{O}, Q \in \partial\Omega$, $0 < R < \min\{\beta/4, R_0\}$ and $u \in W^{1,2}(\Omega)$.*

- *If u is a subsolution in Ω, then*

$$\sup u_M^+ \leq C_p \left(\fint_{B_R(Q)} |u_M^+|^p \right)^{\frac{1}{p}}$$

for any $p > 1$, where $M = \sup_{\partial\Omega \cap B_{2R}(Q)} u^+$ and

$$u_M^+ = \begin{cases} \sup\{u_M, M\} & x \in \Omega \\ M & x \notin \Omega \end{cases}$$

- *If u is a supersolution, which is non-negative in $\Omega \cap B_{4R}(Q)$, then*

$$\left(\fint_{B_{2R}(Q)} |u_m^-|^p \right)^{\frac{1}{p}} \leq C_p \inf_{B_R(Q)} u_m^-$$

for any $1 \leq p < \frac{n}{n-2}$ where $m = \inf_{\partial\Omega \cap B_{4R}(Q)} u$ and

$$u_m^-(X) = \begin{cases} \inf\{u(X), m\} & x \in \Omega \\ m & x \notin \Omega \end{cases}$$

- *If u is a weak solution which is non-negative on $\Omega \cap B_{4R}(Q)$, then*

$$\sup u_M^+ \leq C_q \left(\fint_{B_R(Q)} |u_M^+|^q \right)^{\frac{1}{q}}$$

for all $q > 0$.

Proof. We will prove the subsolution and supersolution results first and, at the end, will mention the changes for the result about weak solutions.
By a scaling argument, we can assume that $R = 1$. Let $\psi \in C_0^1(B_4(Q))$ and $\bar{u} = u_M^+$, if u is a subsolution, and $\bar{u} = u_m^-$, if u is a supersolution. For $\beta \in \mathbb{R}\backslash\{0\}$, we define

$$v = \begin{cases} \psi^2(\bar{u}^\beta - M^\beta) & \text{if } \beta > 0 \\ \psi^2(\bar{u}^\beta - m^\beta) & \text{if } \beta < 0. \end{cases}$$

In the whole proof, β remains away from zero and the case $\beta > 0 (< 0)$ will be applied to u as a subsolution (supersolution) only. Lemma 2.1.6 implies $v \in W_0^{1,2}(\Omega)$ with $v \geq 0$, and therefore v is a valid test function.
We claim that

$$\int_\Omega |\nabla \bar{u}|^2 \psi^2 \bar{u}^{\beta-1} \leq C \int_\Omega (\psi^2 + |\nabla\psi|^2) \bar{u}^{\beta+1}, \tag{2.5}$$

which is (8.52) in [GT01]. To see this, we test with v and get

$$\begin{aligned}
\mathcal{L}(u,v) &= +\int_\Omega A\nabla u\cdot\nabla\bar{u}\,\beta\bar{u}^{\beta-1}\psi^2\\
&\quad+\int_\Omega 2A\nabla u\cdot\nabla\psi\,\psi(\bar{u}^\beta-\tilde{M}^\beta)\\
&\quad+\int_\Omega u\,B\nabla(\psi^2(\bar{u}^\beta-\tilde{M}^\beta))\\
&\quad-\int_\Omega C\nabla u\,\psi^2(\bar{u}^\beta-\tilde{M}^\beta)\\
&\quad-\int_\Omega Du\psi^2(\bar{u}^\beta-\tilde{M}^\beta)\\
&= I+\cdots+V\\
&\leq 0 \text{ for } u \text{ a subsolution and } \tilde{M}=M\\
&\geq 0 \text{ for } u \text{ a supersolution and } \tilde{M}=m.
\end{aligned}$$

Using ellipticity and the sign of β, we get

$$\lambda|\beta|\int_\Omega|\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2\leq II+III+IV+V.$$

The claim (2.5) is proven once we have shown that each term $II,\cdots,V$ can be bounded by $C(1+|\beta|)\int_\Omega(\psi^2+|\nabla\psi|^2)\bar{u}^{\beta+1}+(\varepsilon+C_\varepsilon\varepsilon_1)(1+|\beta|)\int_\Omega|\nabla u|^2\bar{u}^{\beta-1}\psi^2$ for ε and ε_1 small. Roughly speaking, this follows from Lemma (2.1.2), $\bar{u}^\beta-\tilde{M}^\beta\leq\bar{u}^\beta$ and the fact that u can be replaced by $\bar{u}$ in every term. To make it precise, let us start with II:

$$\begin{aligned}
II &\leq 2M\int_\Omega(|\nabla\bar{u}|\bar{u}^{\frac{\beta-1}{2}}\psi)(|\nabla\psi|\bar{u}^{\frac{\beta+1}{2}})\\
&\leq\varepsilon\int_\Omega|\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2+C_\varepsilon\int_\Omega|\nabla\psi^2|\bar{u}^{\beta+1},
\end{aligned}$$

therefore II is done. For IV we use Young's inequality with $\varepsilon(1+|\beta|)$ and Lemma 2.1.2 to get

$$\begin{aligned}
IV &\leq\int_\Omega\left(|\nabla\bar{u}|\bar{u}^{\frac{\beta-1}{2}}\psi\right)\left(|C|\psi(\bar{u}^\beta-\tilde{M}^\beta)\bar{u}^{-\frac{\beta-1}{2}}\right)\\
&\leq\varepsilon(1+|\beta|)\int_\Omega|\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2+\frac{C_\varepsilon\varepsilon_1}{1+|\beta|}\int_\Omega|\nabla(\psi[\bar{u}^\beta-\tilde{M}^\beta]\bar{u}^{\frac{\beta-1}{2}})|^2\\
&\leq\varepsilon(1+|\beta|)\int_\Omega|\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2+\frac{C_\varepsilon\varepsilon_1}{1+|\beta|}\int_\Omega|\nabla\psi|^2\bar{u}^{\beta+1}\\
&\quad+\frac{C_\varepsilon\varepsilon_1}{1+|\beta|}\int_\Omega\psi^2(|\nabla\bar{u}|^2\beta^2\bar{u}^{2(\beta-1)})\bar{u}^{-(\beta-1)}\\
&\quad+\frac{C_\varepsilon\varepsilon_1}{1+|\beta|}\int_\Omega\psi^2(\bar{u}^\beta-\tilde{M}^\beta)^2(|\nabla u|^2\bar{u}^{(-\frac{\beta-1}{2})2}(\frac{\beta-1}{2})^2)\\
&\leq(1+|\beta|)(\varepsilon+C_\varepsilon\varepsilon_1)\int_\Omega|\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2+\frac{C_\varepsilon\varepsilon_1}{1+|\beta|}\int_\Omega|\nabla\psi|^2\bar{u}^{\beta+1},
\end{aligned}$$

which completes term IV. For V, we write $V=\int_\Omega(\bar{u}^{\frac{\beta+1}{2}}\psi)(D\psi(\bar{u}^\beta-\tilde{M}^\beta)\bar{u}^{-\frac{\beta-1}{2}})$. So V can be bounded in the same way as IV. For III, we have

$$\begin{aligned}
III &= \int_\Omega\bar{u}\,B\nabla(\psi^2(\bar{u}^\beta-\tilde{M}^\beta))\\
&= \int_\Omega B\nabla(\bar{u}\psi^2(\bar{u}^\beta-\tilde{M}^\beta))-\int_\Omega B\nabla\bar{u}\,\psi^2(\bar{u}^\beta-\tilde{M}^\beta)=III_A+III_B.
\end{aligned}$$

For III_B, one proceeds as for IV. For III_A, we get

$$
\begin{aligned}
|III_A| &\le C\int_\Omega |\nabla(\bar{u}\psi^2(\bar{u}^\beta - \tilde{M}^\beta))| \\
&\le +C\int_\Omega (|\nabla\bar{u}|\psi\bar{u}^{\frac{\beta-1}{2}})(\psi\bar{u}^{\frac{\beta+1}{2}}) \\
&\quad + C\int_\Omega (|\nabla\psi|\bar{u}^{\frac{\beta+1}{2}})(\psi\bar{u}^{\frac{\beta+1}{2}}) \\
&\quad + C\int_\Omega (\beta^{\frac12}|\nabla\bar{u}|\psi\bar{u}^{\frac{\beta-1}{2}})(\beta^{\frac12}\psi\bar{u}^{\frac{\beta+1}{2}}) \\
&\le \varepsilon(1+|\beta|)\int_\Omega |\nabla\bar{u}|^2\bar{u}^{\beta-1}\psi^2 + C_\varepsilon(1+|\beta|)\int_\Omega(\psi^2+|\nabla\psi|^2)\bar{u}^{\beta+1},
\end{aligned}
$$

which completes claim (2.5). Observe that a smaller ε_1 allows a smaller $|\beta|$. Let $w = \bar{u}^{\frac{\beta+1}{2}}, \beta \neq -1$ and $w = \log\bar{u}$ for $\beta = -1$. Then (2.5) with $\gamma = \beta + 1$ (where γ has to stay away from 1) says

$$
\int_\Omega |\psi\nabla w|^2 \le \begin{cases} C\gamma^2\int_\Omega(\psi^2+|\nabla\psi|^2)w^2 & \text{for } \beta \neq -1 \\ C\int_\Omega \psi^2+|\nabla\psi|^2 & \text{for } \beta = -1 \end{cases} \tag{2.6}
$$

From the Poincaré inequality, we get the following for $\chi = \frac{n}{n-2}$:

$$
||\psi w||_{L^{2\chi}} \le C||(|\nabla\psi|w)||_{L^2} + ||(\psi|\nabla w|)||_{L^2} \le C(1+|\gamma|)||(\psi+|\nabla\psi|)w||_{L^2}.
$$

Thus for $\psi \equiv 1$ on $B_{r_1}(Q)$ and $\psi \equiv 0$ on $B^c_{r_2}(Q)$, with $1 \le r_1 < r_2 \le 3$ and $|\nabla\psi| \le \frac{C}{r_2-r_1}$, we have

$$
||w||_{L^{2\chi}(B_{r_1})} \le C\frac{1+|\gamma|}{r_2-r_1}||w||_{L^2(B_{r_2})}. \tag{2.7}
$$

For $p \neq 0$, define $\Phi(p,r) = (\int_{B_r(Q)}|\bar{u}|^p)^{\frac1p}$, then by Lemma A.0.25 in the Appendix we get $\lim_{p\to\infty}\Phi(p,r) = \sup_{B_r(Q)}\bar{u}$ and $\lim_{p\to-\infty}\Phi(p,r) = \inf_{B_r(Q)}\bar{u}$. By the definition of w and (2.7), we have $\Phi(\gamma\chi,r_1)^{\frac{\gamma}{2}} \le \frac{C(1+|\gamma|)}{r_2-r_1}\Phi(\gamma,r_2)^{\frac{\gamma}{2}}$, hence

$$
\Phi(\chi\gamma,r_1) \le \left(\frac{C(1+|\gamma|)}{r_2-r_1}\right)^{\frac{2}{|\gamma|}}\Phi(\gamma,r_2) \text{ for } \gamma > 0 \tag{2.8}
$$

$$
\Phi(\gamma,r_2) \le \left(\frac{C(1+|\gamma|)}{r_2-r_1}\right)^{\frac{2}{|\gamma|}}\Phi(\chi\gamma,r_1) \text{ for } \gamma < 0 \tag{2.9}
$$

We can start the iteration on k with $\gamma = p\chi^{m-k}$ and $r^k = 1 + \frac{1}{2^{m-k}} - \frac{1}{2^m}, k = 0, \cdots, m$ to get

$$
\begin{aligned}
\Phi(\chi^{m+1}p, r^0) &\le \left(\frac{C(1+p\chi^m)}{2^{-(m+0)}}\right)^{\frac{2}{\chi^m p}}\Phi(\chi^m p, r^1) \\
&\le \prod_{k=0}^m \left(\frac{C(1+p\chi^{m-k})}{2^{-(m-k)}}\right)^{\frac{2}{\chi^{m-k}p}}\Phi(p,r^m).
\end{aligned}
$$

The term $\Phi(p,r^m)$ is bounded by $\Phi(p,2)$. The product of terms is bounded by a constant, since for example

$$
\prod_{k=0}^m 2^{\frac{2(m-k)}{\chi^{m-k}p}} = 2^{\frac2p\sum_{k=0}^m k\chi^{-k}} \le C
$$

Thus we see that $\Phi(\chi^m p, 1) \le C\Phi(p,2)$. Sending $m \to \infty$, we get the desired result for subsolutions, whereby the closer p gets to 1, the closer γ gets to 1, i.e. the smaller ε_1 has to be.

For u a supersolution, we can use the same method of iteration to show that for fixed $0 < p_0 < p < \chi$ we get

$$\Phi(p,2) \le C\Phi(p_0,3) \text{ by (2.8) for } 0<\gamma<1,$$
$$\Phi(-p_0,3) \le C\Phi(-\infty,1) \text{ by (2.9)}.$$

It remains necessary to show that there exists $0 < p_0 < \min\{\chi, p\}$ such that $\Phi(p_0,3) \le \Phi(-p_0,3)$. Using (2.6) for $\beta = -1$ we get with $\Psi \equiv 1$ on $B_r(Q), \Psi \equiv 0$ on $B_{2r}(Q)$ and $|\nabla\Psi| \le C/r$ that $\int_{B_r} |\nabla w| \le Cr^{n-1}$. Theorem A.0.27 in the Appendix implies the existence of a $p_0 > 0$ such that

$$\int_{B_3(Q)} e^{p_0|w-w_{B_3(Q)}|} \le C.$$

Hence $\int_{B_3(Q)} e^{p_0 w} \int_{B_3(Q)} e^{-p_0 w} \le Ce^{p_0 w_{B_3(Q)}} e^{-p_0 w_{B_3(Q)}} = C$ and therefore, since $w = \log \bar{u}$, we get $\Phi(p_0,3) \le C\Phi(-p,3)$ and so the supersolution result is proven.
For the result about weak solutions, let $\bar{u} = u_M^+$ and $v = \psi(\bar{u}^\beta - M^\beta)$ for $\beta \neq 0$. Then, as before, (2.5) holds for a constant C uniformly in β, if $|\beta|$ stays away from zero. Therefore (2.8) holds for $\gamma > 0$, whereby γ has to stay away from 1.
One can apply the same iteration as for subsolutions. In the case that there exists an $m \in \mathbb{N}$ such that $q\chi^m = 1$, one has to choose a slightly smaller q. Moreover, the smaller q is the smaller ε_1 has to be, since $\sup_{0<q_0<q} \inf_m |q_0\chi^m - 1| \to 0$ as $q \to 0$. $\square$

With Theorem 2.2.7 proven one can follow [GT01] to show that the continuous Dirichlet problem is solvable. Since we did not prove Theorem 2.2.7 in such a generality as D. Gilbarg and N.S. Trudinger in [GT01], we have to work a little bit more carefully.

Theorem 2.2.8. *Assume that u is a weak solution in the Lipschitz domain Ω for $L \in \mathcal{O}$, then for any $0 < 2R < \delta \le \min\{R_0, \beta\}$, we have*

$$osc_{T_R(Q)} u \le C\left[\left(\frac{R}{\delta}\right)^\alpha \sup_{T_\delta(Q)} |u| + \sigma(\sqrt{R\delta})\right],$$

where $\sigma(R) = osc_{\Delta_R(Q)} u$ and $osc_{T_R(Q)} u = |\sup_{T_R(Q)} u - \inf_{T_R(Q)} u|$ and $osc_{\Delta_R(Q)}$ accordingly.

Proof. We follow the proof of Theorem 8.27 in [GT01] and include a few lines since we did not prove Theorem 2.2.7 in such a generality as in [GT01]. We define the following numbers:

$$M_1 = \sup_{T_R(Q)} u, \quad M_4 = \sup_{T_{4R}(Q)} u, \quad M = \sup_{\partial\Omega\cap B_{4R}(Q)} u,$$
$$m_1 = \inf_{T_R(Q)} u, \quad m_4 = \inf_{T_{4R}(Q)} u, \quad m = \inf_{\partial\Omega\cap B_{4R}(Q)} u.$$

We will consider first the case if $M_4 \ge 0$ and $m_4 \le 0$. By the assumptions on $L \in \mathcal{O}$, we have that positive constants are supersolutions. Thus the functions $M_4 - u$ and $u - m_4$ are supersolutions. As in Theorem 2.2.7, we define

$$(u-m_4)^-_{m-m_4} = \begin{cases} \inf\{u-m_4, m-m_4\} & x \in \Omega \\ m - m_4 & x \notin \Omega, \end{cases}$$

and similarly for $(M_4 - u)^-_{M_4-M}$. Using this definition for the first inequality in each of the following two lines and then the result about supersolutions in Theorem 2.2.7 for the second inequality, we get

$$(M_4 - M)\frac{|B_{2R}(Q)\backslash\Omega|}{R^n} \le \frac{1}{R^n}\int_{B_{2R}(Q)} (M_4-u)^-_{M_4-M} \qquad \le C(M_4 - M_1),$$
$$(m - m_4)\frac{|B_{2R}(Q)\backslash\Omega|}{R^n} \le \frac{1}{R^n}\int_{B_{2R}(Q)} (u-m_4)^-_{m-m_4} \qquad \le C(m_1 - m_4).$$

Since Ω is a Lipschitz Domain, it satisfies a uniform exterior cone condition and so $|B_{2R}(Q)\backslash\Omega| \geq CR^n$ for a uniform constant $C = C(\Omega)$. Thus the two inequalities above imply

$$M_4 - M \leq C(M_4 - M_1),$$
$$m - m_4 \leq C(m_1 - m_4).$$

Adding them together, we are left with

$$M_1 - m_1 \leq (1 - \frac{1}{C})(M_4 - m_4) + \frac{1}{C}(M - m),$$

which implies that

$$\text{osc}_{T_R(Q)} u \leq \gamma \, \text{osc}_{T_{4R}(Q)} u + C \, \text{osc}_{\Delta_{4R}(Q)} u$$

for some $0 < \gamma < 1$. The theorem in the case that $M_4 \geq 0$ and $m_4 \leq 0$ then follows from Lemma A.0.26 in the Appendix.
In the other case we can assume without losing generality that $0 < m_4 \leq M_4$. As before we then consider $M_4 - u$ and $u - m_4$. In order to repeat the previous argument it remains to show that the result about supersolutions from the previous theorem can be applied to $u - m_4$, which is a subsolution. For this it is enough to show that (2.5) holds for $\beta < 0$ Therefore we define v and $\tilde{M}$ as in the previous proof for supersolutions, then

$$\mathcal{L}(u - m_4, v) = I' + II' + III' + IV' = \underbrace{-m_4 \int_\Omega B\nabla(\psi^2(\bar{u}^\beta - \tilde{M}^\beta))}_{V'} + \underbrace{m_4 \int_\Omega D\psi^2(\bar{u}^\beta - \tilde{M}^\beta)}_{VI'}.$$

As in the previous proof we use ellipticity to get

$$\lambda|\beta| \int_\Omega |\nabla \bar{u}|^2 \bar{u}^{\beta-1} \psi^2 \leq |II'| + \cdots + |VI'|.$$

Since $|u - m_4| \leq u$ the terms II', III', IV' are treated as the terms II, III, IV. For V' and VI' observe that $m_4 \leq \bar{u}$ and so for example

$$m_4 \int_\Omega |\nabla(\psi^2(\bar{u}^\beta - \tilde{M}^\beta))| \leq \int_\Omega \bar{u}|\nabla(\psi^2(\bar{u}^\beta - \tilde{M}^\beta))|.$$

Therefore, one can proceed as for the terms II, III, IV to bound the terms IV' and VI'. Thus (2.5) is proven for $u - m_4$ and so one can apply the supersolution result of Theorem 2.2.7 for the subsolution $u - m_4$. One proceeds as in the case that $m_4 \leq 0$ and $M_4 \geq 0$ to finish the proof. □

Remark 2.2.9. *The previous theorem tells us that if $\sigma(R) \to 0$ as $R \to 0$, then*

$$\lim_{\Omega \ni X \to Q} u(X) = u(Q)$$

is well defined, i.e. we can extend u to $\bar{\Omega}$ in a continuous way.

Lemma 2.2.10. *Let $L \in \mathcal{O}$ and Ω be a Lipschitz domain. Assume that u is a weak solution and $osc_{\Delta_R(Q)} u \to 0$ as $R \to 0$ for all $Q \in \partial\Omega$, then u is uniformly continuous on $\bar{\Omega}$.*

Proof. The proof is obvious since continuous functions on a compact set are always uniformly continuous. Nevertheless, we will formulate a proof, which will provide a further result.
Fix $\varepsilon > 0$. Since Ω is a Lipschitz domain, Ω satisfies a uniform cone condition at every $Q \in \partial\Omega$. Let $M = \sup_\Omega |u|$. By Theorem 2.2.8, we see that there exists $\beta_1 > 0$ depending on M and the constants C and α in Theorem 2.2.8 such that

$$\text{osc}_{T_{\beta_1}(Q)} u \leq \varepsilon$$

for all $Q \in \partial\Omega$, i.e. u is uniformly continuous on $\overline{(\partial\Omega)}_{\beta_1/2}$.
It remains to be shown that $\text{osc}_{B_\gamma(X)} u \leq \varepsilon$ for some $\gamma < \beta_1/2$ and all $X \in \Omega_{\beta_1/2}$. Weak solutions

are Hölder continuous on $\Omega_{\beta_1/2}$. Thus, there exists α and C such that $|u(X) - u(Y)| \leq C|X - Y|^\alpha$. For $|X-Y| < \gamma$ with γ small enough, depending on C, β_1 and α we get $|u(X)-u(Y)| < \varepsilon$. Hence

$$\operatorname{osc}_{B_\gamma(\tilde{X})\cap\bar{\Omega}} u \leq \varepsilon$$

for all $\tilde{X} \in \bar{\Omega}$, where γ can be chosen in a way that it depends on ε and the constants in the definition of the operator class $\mathcal{O}$, but is independent of u. $\square$

Remark 2.2.11. *Following the proof of Lemma 2.2.10, we see that a uniformly bounded sequence of weak solutions u_k corresponding to a sequence of elliptic operators $L_k \in \mathcal{O}$ is equicontinuous on $\bar{\Omega}$.*

Finally, we can show that the continuous Dirichlet Problem for elliptic operators in $\mathcal{O}$ is solvable:

Theorem 2.2.12. *Let $L \in \mathcal{O}$ and Ω be a Lipschitz domain. For every $g \in C^0(\partial\Omega)$, there exists a unique $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ such that $Lu = 0$ in Ω and $u \equiv g$ on $\partial\Omega$.*

Proof. Choose $\{\varphi_m\} \in C^\infty(\partial\Omega)$, which converge uniformly to g. Let u_m be the weak solution to $Lu_m = 0$ in Ω and $u_m = \varphi_m$ on $\partial\Omega$ (existence is guaranteed by Theorem 2.2.6). Define $\alpha_{mk} = \sup_{\partial\Omega} |\varphi_m - \varphi_k|$ (we also denote the weak solution with the constant boundary value α_{mk} by α_{mk}), then $lim_{X\to Q} u_m - u_k - \alpha_{mk} \leq 0$ for all Q, hence by the maximum principle $u_m - u_k \leq \alpha_{mk}$. The same holds for $u_k - u_m$ and therefore

$$\sup_\Omega |u_m - u_k| \leq \sup_{\partial\Omega} |\varphi_m - \varphi_k| \to 0.$$

So u_m converges uniformly to some $u \in C^0(\bar{\Omega})$ with $u \equiv \varphi$ on $\partial\Omega$. In addition, by the interior Caccioppoli estimate we get $\int_{\Omega'} |\nabla(u_m - u_k)|^2 \to 0$ for all compact $\Omega' \subset \Omega$. The uniqueness of limits implies $u \in W^{1,2}_{loc}(\Omega)$, and since the coefficients are in $L^\infty(\Omega')$, one sees that u is a weak solution.
Uniqueness follows from the maximum principle. $\square$

We will finish this section with a result about subsolutions from [Sta65] for our type of elliptic operators. The proof given in [Sta65] works equally well for $L \in \mathcal{O}$. For completeness, we will include this proof from [Sta65], which is based on the following Hilbert space result:

Corollary 2.2.13 (Corollary 2.1 in [Sta65])**.** *Let $B(\cdot,\cdot)$ be a bilinear form on $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ such that for fixed $g \in W^{1,2}(\Omega)$, $B(g,\cdot)$ is continuous on $W^{1,2}_0(\Omega)$ and $B(\cdot,\cdot)$ is coercive on $W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega)$.*
Let $U \subset W^{1,2}(\Omega)$ be a convex and closed subset with $Tr(u) = Tr(g)$ for all $u \in U$. Then, for a fixed $f \in (W^{1,2}(\Omega))^*$, there exists a unique $u \in U$ such that

$$B(u, v) \geq \langle f, v\rangle$$

for all $v \in V_u = \{v \in W^{1,2}(\Omega) :$ it exists $\varepsilon > 0$ such that $u + \varepsilon v \in U\}$.

Theorem 2.2.14. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}$. Assume that u and v are subsolutions, then $w = \max\{u, v\}$ is a subsolution.*

Proof. We follow the proof of Theorem 3.5 in [Sta65]. Let $U = \{\vartheta \in W^{1,2}(\Omega) : Tr(\vartheta) = Tr(w), \vartheta \leq w\}$. Then U is closed and convex. Let $\mathcal{L}_\sigma$ be the bilinear form corresponding to $L_\sigma, \sigma > 0$, which was defined in the proof of Theorem 2.2.5. Then $\mathcal{L}_\sigma$ is coercive on $W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega)$ and so by Corollary 2.2.13, there exists a unique $\eta \in U$ such that

$$\mathcal{L}_\sigma(\eta, \varphi) \geq \sigma \int_\Omega w\varphi \tag{2.10}$$

for all $\varphi \in V_\eta$. Since all non-positive $\psi \in C^\infty_0(\Omega)$ are in V_η we get for all non-negative $\varphi \in C^\infty_0(\Omega)$ since $\eta \leq w$ by the definition of U that

$$\mathcal{L}(\eta, \varphi) \leq \int_\Omega \sigma(w - \eta)\varphi \leq 0.$$

This says that η is a subsolution. We claim that $w \leq \eta$ which clearly finishes the proof, since it implies $\eta = w$. To see this, let $\zeta = \max\{u, \eta\}$, then $\zeta \in U$. We will show that $\zeta = \eta$. Since $\zeta \in U$, we get $\zeta - \eta \in V_\eta$ (choose $\varepsilon = 1$) and so, by (2.10),

$$\mathcal{L}_\sigma(\eta, \zeta - \eta) \geq \sigma \int_\Omega w(\zeta - \eta).$$

Since u is a subsolution and $\zeta - \eta = 0$ on the set where $\eta \geq u$, we get

$$\mathcal{L}(\zeta, \zeta - \eta) = \mathcal{L}(u, \zeta - \eta) \leq 0.$$

Therefore, $\mathcal{L}_\sigma(\zeta, \zeta - \eta) \leq \sigma \int_\Omega \zeta(\zeta - \eta)$. Thus

$$\mathcal{L}_\sigma(\zeta - \eta, \zeta - \eta) \leq \sigma \int_\Omega (\zeta - w)(\zeta - \eta) \leq 0,$$

since $\zeta - \eta \geq 0$ by the definition of ζ and $\zeta - w \leq 0$ by the definition of U. Therefore, $\zeta = \eta$ and so $u \leq \eta$. The same reasoning shows that $v \leq \eta$ and so $w \leq \eta$. □

2.3 Behaviour of weak solutions at the Boundary

In this section, we will study the behaviour of weak solutions, which vanish at a part of the boundary. The first result we get is that weak solutions of that kind are viable test functions for that part of the boundary.

Lemma 2.3.1. *Let $L \in \mathcal{O}$ and Ω be a Lipschitz domain. Assume that $u \in W^{1,2}_{loc}(T_{2R}(Q)) \cap C(\overline{T_{2R}(Q)})$ is a weak solution or a non-negative subsolution for $L \in \mathcal{O}$ on $T_{2R}(Q)$. In addition, assume that u vanishes on $\Delta_{2R}(Q)$. Then $u \in W^{1,2}(T_R(Q))$.*

Proof. We modify the proof for weak solutions to elliptic operators of the form $\mathrm{div} A\nabla u$ in the remark 1.2 of [CFMS81] for our type of elliptic operators.
Without losing generality, we can assume that $R \leq \min\{R_0, \beta\}$. For $s > 0$ let $\psi = [u - s]_+ \phi^2$, where $\phi \in C^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ on $B_R(Q)$ and 0 outside of $B_{3R/2}(Q)$ (we mutually extend u to be 0 outside of $T_{2R}(Q)$). Then ψ is non-negative and in $W^{1,2}_0(T_{2R}(Q))$. Hence it is a viable test function and so

$$\begin{aligned} 0 &\geq \mathcal{L}(u, \psi) \\ &= \int_\Omega A\nabla u \cdot \nabla([u - s]_+\phi^2) + u\, B\nabla([u - s]_+\phi^2) - C\nabla u\, [u - s]_+\phi^2 - Du[u - s]_+\phi^2. \end{aligned}$$

Using the product rule in the term involving A, ellipticity and Young's inequality with ε, we get that

$$\begin{aligned} \int_{\{u>s\}} |\nabla u|^2\phi^2 \leq C \int_{\{u>s\}} |\nabla \phi|^2 u^2 &+ C\Big|\int_{\{u>s\}} u\, B\nabla([u - s]_+\phi^2)\Big| \\ &+ C\Big|\int_{\{u>s\}} C\nabla u\, [u - s]_+\phi^2\Big| + C\Big|\int_{\{u>s\}} Du[u - s]_+\phi^2\Big|. \end{aligned}$$

We claim that the terms involving B, C, D are bounded by $C \int_{T_{2R}(Q)} |\nabla \phi|^2 u^2 + \frac{1}{10} \int_{\{u>s\}} |\nabla u|^2\phi^2$.

We show this exemplarily only for the term involving B:

$$
\begin{aligned}
|\int_{\{u>s\}} u\, B\nabla([u-s]_+\phi^2)| &\le |\int_{\{u>s\}} B\nabla(u[u-s]_+\phi^2)| + |\int_{\{u>s\}} B\nabla u\,[u-s]_+\phi^2| \\
&\le C\int_{\{u>s\}} |\nabla(u[u-s]_+\phi^2)| + \varepsilon\int_{\{u>s\}} |\nabla u|^2\phi^2 \\
&\quad + C_\varepsilon \int_{\{u>s\}} |B|^2[u-s]_+^2\phi^2 \\
&\le (\varepsilon + C_\varepsilon\varepsilon_1)\int_{\{u>s\}} |\nabla u|^2\phi^2 + C_\varepsilon \int_{T_{2R}(Q)} u^2(\phi^2+|\nabla\phi|^2),
\end{aligned}
$$

where we used Young's inequality with $\varepsilon > 0$. Treating the other terms in a similar manner and choosing $\varepsilon > 0$ sufficiently small, we are left with

$$\int_{\{u>s\}} |\nabla u|^2\phi^2 \le C\int_{T_{2R}(Q)} (|\nabla\phi|^2+\phi^2)u^2.$$

The monotone convergence theorem shows that $u^+ \in W^{1,2}(T_R(Q))$. Thus the proof is complete, if u is a non-negative subsolution. In case u is a solution, we use the fact that u^+ and $(-u)^+$ are by Theorem 2.2.14 subsolutions. One can repeat the proof for u^+ and $(-u)^+$ to conclude that $u \in W^{1,2}(T_R(Q))$. □

Lemma 2.3.2 (Cacciopoli's inequality at the boundary)**.** *Let $u \in W^{1,2}_{loc}(T_{2R}(Q)) \cap C^0(\overline{T_{2R}(Q)})$ be a non-negative subsolution for $L \in \mathcal{O}$ on the Lipschitz domain Ω, which vanishes on $\Delta_{2R}(Q)$, then*

$$\int_{T_R(Q)} |\nabla u|^2 \le \frac{C}{R^2}\int_{T_{2R}(Q)} u^2.$$

Proof. We repeat the proof of Lemma 2.3.1 with $\psi = u\phi^2$, where $0 \le \phi \le 1$ with $\phi \equiv 1$ on $B_R(Q)$, $\phi \equiv 0$ on $B_{2R}(Q)^c$ and $|\nabla\phi| \le \frac{C}{R}$. Then ψ is a viable test function by Lemma 2.3.1 and so

$$\int_{T_{2R}(Q)} |\nabla u|^2\phi^2 \le C\int_{T_{2R}(Q)} (|\nabla\phi|^2+\phi^2)u^2,$$

which completes the proof. □

In [HL01] (Lemma 3.9), the Hölder Continuity up to the boundary for weak solutions was proven for elliptic operators of the form $\mathrm{div}A\nabla + C\nabla$, with A and C as in the definition of $\mathcal{O}$ and smooth. Since we have already proven that the continuous Dirichlet problem for $L \in \mathcal{O}$ is solvable, the same proof can be extended to $L \in \mathcal{O}$.

Lemma 2.3.3. *Let Ω be a Lipschitz domain and u be a non-negative weak solution to $L \in \mathcal{O}$, which vanishes on $\Delta_{2R}(Q)$. Then, for $X \in T_R(Q)$, we have*

$$u(X) \le C\left(\frac{\delta(X)}{R}\right)^\alpha u(A_R(Q)).$$

Proof. By the Harnack principle, we can assume that $R \le \min\{R_0, \beta\}$ and that it is enough to show

$$\fint_{T_{\rho/4}(Q)} u^2 \le C\left(\frac{\rho}{R}\right)^\alpha u(A_R(Q))^2$$

for $\rho < R/4$. For a given τ with $\rho/2 \le \tau \le \rho/8$, let u_0 be the solution to $L_0 = \mathrm{div}A\nabla$ on $T_\tau(Q)$ with boundary values u. Existence is guaranteed by Theorem 2.2.12 since $u \in C^0(\overline{T_\tau(Q)})$. Define $w = u - u_0$, then $w \in W^{1,2}_0(T_\tau(Q))$ and so

$$
\begin{aligned}
0 &= \mathcal{L}(u,w) \\
0 &= \mathcal{L}_0(u_0,w).
\end{aligned}
$$

Subtracting the second equality from the first gives us

$$\begin{aligned}\int_{T_\tau(Q)} |\nabla w|^2 &\leq C|\int_{T_\tau(Q)} u\, B\nabla w| + C|\int_{T_\tau(Q)} C\nabla u\, w| + C|\int_{T_\tau(Q)} Duw| \\ &\leq (\varepsilon + C_\varepsilon \varepsilon_1)\int_{T_\tau(Q)} |\nabla w|^2 + (\varepsilon + C_\varepsilon\varepsilon_1)\int_{T_\tau(Q)} |\nabla u|^2\end{aligned}$$

by Poincaré's inequality. So for ε_1 small enough we get

$$\frac{1}{\tau^{n-2}}\int_{T_\tau(Q)} |\nabla w|^2 \leq \bar{\varepsilon}\frac{1}{\tau^{n-2}}\int_{T_\tau(Q)} |\nabla u|^2, \tag{2.11}$$

for $\bar{\varepsilon}$ small, where $\bar{\varepsilon}$ can be chosen in a way that $\bar{\varepsilon} \searrow 0$ for $\varepsilon_1 \searrow 0$. Let $\Phi(f,\nu) = \frac{1}{\nu^{n-2}}\int_{T_\nu(Q)} |\nabla f|^2$. Using the boundary Hölder continuity, the boundary Cacciopoli estimate and Poincaré's inequality for solutions to L_0-type operators, we get

$$\Phi(u_0,\nu) \leq C\left(\frac{\nu}{\tau}\right)^{2\alpha'}\Phi(u_0,\tau).$$

This and (2.11) imply for $0 < \nu < \tau$:

$$\Phi(u,\nu) = \Phi(w + u_0,\nu) \leq C\left[\left(\frac{\nu}{\tau}\right)^{2\alpha'} + \bar{\varepsilon}\left(\frac{\tau}{\nu}\right)^{n-2}\right]\Phi(u,\tau).$$

Set $\theta = \frac{\nu}{\tau}$. We can choose first ν small and then ε_1 (and so $\bar{\varepsilon}$) small enough, to get

$$\Phi(u,\theta\tau) \leq \frac{1}{2}\Phi(u,\tau).$$

Iterating this, starting at $\tau = \frac{R}{8}$ and ending at $\frac{\rho}{2}$, we get $\Phi(u,\frac{\rho}{2}) \leq \left(\frac{1}{2}\right)^k \Phi(u,\frac{R}{8})$, where $\theta^{k+1}\frac{R}{8} = \frac{\rho}{2}$. So $k = \frac{\log_2(\frac{\rho}{R}4)}{\log_2(\theta)}$, hence for $2\alpha = -\frac{1}{\log_2\theta}$ we get

$$\Phi(u,\frac{\rho}{2}) \leq C\left(\frac{\rho}{\nu}\right)^{2\alpha}\Phi(u,\frac{r}{8}).$$

Using the Poincaré inequality on the left and the Cacciopoli inequality on the right gives

$$\fint_{T_{\rho/2}(Q)} |u|^2 \leq C\left(\frac{\rho}{\nu}\right)^{2\alpha}\fint_{T_\nu(Q)} |u|^2 \leq C\left(\frac{\rho}{\nu}\right)^{2\alpha}\max_{T_\nu(Q)} u^2(X). \tag{2.12}$$

The following Lemma, which is a consequence of the Harnack inequality and (2.12), finishes the proof. The proof is taken from the proof in [Bau84] of Lemma 2.4. ☐

Lemma 2.3.4. *Let L,Ω,Q,R and u be as above, then $u(X) \leq Cu(A_R(Q))$ for all $X \in T_R(Q)$.*

Proof. Without losing generality, we can assume that $u(A_R(Q)) = 1$. Harnack's inequality says that there exist $c_1 > 2, c_2 > 1$ such that $\max_{B_{\frac{1}{c_1}\delta(X)}(X)} u \leq c_2 \min_{B_{\frac{1}{c_1}\delta(X)}(X)} u$. Hence if $u(Y) > c_2^h$, the Harnack inequality implies $\delta(Y) \leq c_1^{-h}R$. For $X \in T_{c_1^{-1}s}(P)$ with c_1 large, we have, by (2.12),

$$u(X) \leq C\fint_{B_{\frac{\delta(X)}{2}}(X)} u \leq C\fint_{T_{\delta(X)}(\hat{X})} u \leq C\left(\frac{\delta(X)}{s}\right)^\alpha \max_{T_{s/2}(\hat{X})} u \leq Cc_1^{-\alpha}\max_{T_s(P)} u.$$

Thus we can choose c_1 large enough such that

$$\frac{1}{2}\max_{X\in T_s(P)} u(X) \geq \max_{X\in T_{c_1^{-1}s}(P)} u(X). \tag{2.13}$$

Let $M \geq 1$ be such that $2^M \geq c_2$ and let $N = M + 5$.
Suppose there exists Y_0 in $T_R(Q)$ with $u(Y_0) \geq c_2^N = c_2^N u(A_R(Q))$, then we claim that there is a sequence $\{Y_k\} \in T_R(Q)$ with $\delta(Y_k) \to 0$ and $u(Y_k) \to \infty$, which contradicts the assumption that $u \equiv 0$ on $\Delta_{2R}(Q)$.
The Harnack inequality implies $\delta(Y_0) < c_1^{-N} R$. Hence (2.13) implies

$$\max_{X \in T_{(c_1^{-5} R)}(\hat{Y})} u(X) = \max_{X \in T_{(c_1^{-N+M} R)}(\hat{Y})} u(X) \geq 2^M \max_{X \in T_{(c_1^{-N} R)}(\hat{Y})} u(X) \geq 2^M u(Y_0) \geq c_2^{1+N},$$

i.e. there exists $Y_1 \in T_{(c_1^{-5} R)}(\hat{Y})$ such that $u(Y_1) \geq c_2^{N+1}$. Harnack's inequality implies $\delta(Y_1) \leq c_1^{-N-1} R$. Iterating these steps we obtain sequences Y_k and $\hat{Y}_k$ with

$$\begin{aligned} Y_k \in T_{(c_1^{-5-(k-1)} R)}(\hat{Y}_{k-1}) \subset T_{\frac{3}{2}R}(Q), k \geq 1 \\ \delta(Y_k) \leq c_1^{-N-k} R \\ u(Y_k) \geq c_2^{N+k}, \end{aligned}$$

which finishes the claim and so, the proof is complete. □

2.4 An Approximation Argument

In this section, we will study the question if a sequence of weak solutions u_k corresponding to a sequence of elliptic operators $L_k \in \mathcal{O}$ and some fixed boundary values converge to the weak solution u of the elliptic operator L and the same boundary values, if L_k converges in some sense to L.
In [KP93] and in [KKPT00], it is shown that if $A_j \to A$ almost everywhere, then the weak solutions

$$\begin{aligned} \mathrm{div} A_j \nabla u_j &= 0 \text{ in } \Omega \\ u_j &\equiv \psi \text{ on } \partial\Omega \end{aligned}$$

for $\psi \in W^{1,2}(\Omega)$ converge in $W^{1,2}(\Omega)$ to the weak solution u with $\mathrm{div} A \nabla u = 0$ in Ω and $u \equiv \psi$ on $\partial\Omega$. The proof is based on the fact that operators of the form $\mathrm{div} A \nabla$ are coercive and that solutions $v \in W_0^{1,2}(\Omega)$ to $\mathrm{div} A \nabla v = \mathrm{div}\psi$ for $\psi \in L^\infty$ are in $W_0^{1,p}(\Omega)$ for some $p > 2$.
We will show the same for our type of elliptic operators under the restriction that constant functions are weak solutions. The fact that our bilinear forms $\mathcal{L}(\cdot, \cdot)$ are not coercive will be bypassed by the usage of the Green's operator $\mathcal{G}$.

Lemma 2.4.1. *Let $L_0, L_1 \in \mathcal{O}$ and Ω be a Lipschitz domain. Assume that $u \in W_0^{1,p_1}(\Omega)$ and $v \in W_0^{1,p_2}(\Omega)$ with $rp_1' = p_2$ for some $r > 1$ (i.e. $\frac{1}{p_1} + \frac{1}{p_2} < 1$), then*

$$\begin{aligned} |(\mathcal{L}_0 - \mathcal{L}_1)(u, v)| \leq C(&||A_0 - A_1||_{L^{p_1' r'}(\Omega)} + ||\delta(\cdot)(B_0 - B_1)||_{L^{p_1' r'}((\partial\Omega)_\beta)} \\ &+ ||B_0 - B_1||_{L^{p_1' r'}(\Omega_\beta)} + ||\delta(\cdot)(C_0 - C_1)||_{L^{p_1' r'}((\partial\Omega)_\beta)} \\ &+ ||C_0 - C_1||_{L^{p_1' r'}(\Omega_\beta)} + ||\delta(\cdot)(D_0 - D_1)||_{L^{p_1' r'}((\partial\Omega)_\beta)} \\ &+ ||D_0 - D_1||_{L^{p_1' r'}(\Omega_\beta)})\, ||u||_{W^{1,p_1}(\Omega)}\, ||v||_{W^{1,p_2}(\Omega)}. \end{aligned}$$

Proof. Exemplary we prove the term involving $C_0 - C_1$:

$$\begin{aligned} |\int_\Omega (C_0 - C_1) \nabla u\, v| &\leq ||\nabla u||_{L^{p_1}(\Omega)} \left(||(C_0 - C_1)v||_{L^{p_1'}((\partial\Omega)_\beta)} + ||(C_0 - C_1)v||_{L^{p_1'}(\Omega_\beta)} \right) \\ &\leq C ||\nabla u||_{L^{p_1}(\Omega)} (||(C_0 - C_1)\delta(\cdot)||_{L^{p_1' r'}((\partial\Omega)_\beta)} \\ &\quad + ||C_0 - C_1||_{L^{p_1' r'}(\Omega_\beta)})\, ||v||_{W_0^{1,p_2}(\Omega)} \end{aligned}$$

by Lemma 2.1.2. □

In the following, F_ψ will denote $L\psi = \mathcal{L}(\psi, \cdot)$ for $\psi \in C^\infty(\mathbb{R}^n)$.

Lemma 2.4.2. *Let $L \in \mathcal{O}$ with $B, D \equiv 0$ and Ω be a Lipschitz domain. Assume that $u \in W_0^{1,2}(\Omega)$ (where we consider $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$) satisfies $Lu = F_\psi$ in Ω with $\psi \in C^\infty(\mathbb{R}^n)$. Then, for any $X \in \Omega$ and $R > 0$, there exists $q < 2$ such that*

$$\left(\fint_{\Omega\cap B_R(X)} |\nabla u|^2\right)^{\frac{1}{2}} \leq C\left(\fint_{\Omega\cap B_{2R}(X)} |\nabla u|^q\right)^{\frac{1}{q}} + C\left(\fint_{\Omega\cap B_{2R}(X)} |\nabla \psi|^2\right)^{\frac{1}{2}},$$

where the constant C is independent of R and X.

Proof. We adjust the proof for Lemma 7.1 in [KP93]. Let φ be a cut-off function for $B_R(X)$, i.e. $\varphi \in C_0^\infty(B_{\frac{3}{2}R}(X))$ with values in $[0,1]$ and $\varphi \equiv 1$ on $B_R(X)$, $|\nabla\varphi| \leq \frac{C}{R}$. We define the test function $v = (u - c_0)\varphi^2$, where $c_0 = 0$ if $B_{\frac{3}{2}R}(X) \cap \Omega^c \neq \emptyset$ and $c_0 = \fint_{B_{2R}(X)} u$ if otherwise. Then, $\mathcal{L}(u, v) = F_\psi(v)$, i.e.

$$\int_\Omega A\nabla u \cdot \nabla([u-c_0]\varphi^2) - \int_\Omega C\nabla u\,(u-c_0)\varphi^2 = \int_\Omega A\nabla\psi \cdot \nabla([u-c]_0\varphi^2) - \int_\Omega C\nabla\psi\,(u-c_0)\varphi^2.$$

Ellipticity implies

$$\begin{aligned}\int_\Omega |\nabla u|^2\varphi^2 \leq{}& C_\lambda \int_\Omega |\nabla u||\nabla\varphi||u-c_0|\varphi + C_\lambda \left|\int_\Omega C\nabla u\,(u-c_0)\varphi^2\right| \\ &+ C_\lambda\left|\int_\Omega A\nabla\psi\cdot\nabla([u-c_0]\varphi^2)\right| + C_\lambda\left|\int_\Omega C\nabla\psi\,(u-c_0)\varphi^2\right| \\ ={}& I + II + III + IV.\end{aligned}$$

We claim that each term of I, II, III, IV is dominated by

$$(\varepsilon + C_\varepsilon\varepsilon_1)\int_\Omega |\nabla u|\varphi^2 + C_\varepsilon \int_\Omega |u-c_0|^2|\nabla\varphi|^2 + C\int_\Omega |\nabla\psi|^2\varphi^2. \tag{2.14}$$

For I, one applies Young's inequality with ε to get the bound (2.14). For II, we use Lemma 2.1.2 to get

$$\begin{aligned}II \leq{}& C\left(\left|\int_{(\partial\Omega)_\beta} C\nabla u\,(u-c_0)\varphi^2\right| + \left|\int_{\Omega_\beta} C\nabla u\,(u-c_0)\varphi^2\right|\right) \\ \leq{}& \varepsilon\int_\Omega |\nabla u|^2\varphi^2 + C_\varepsilon\varepsilon_1\int_\Omega |\nabla u|^2\varphi^2 + C_\varepsilon\varepsilon_1\int_\Omega |u-c_0|^2|\nabla\varphi|^2 \\ &+ M\varepsilon\int_\Omega |\nabla u|^2\varphi^2 + MC_\varepsilon\int_\Omega |u-c_0|^2\varphi^2.\end{aligned}$$

Thus (2.14) holds for II. The terms III and IV are proven in a similar way. Therefore, if ε_1 is small enough, we get

$$\left(\fint_{B_R(X)} |\nabla u|^2\right)^{\frac{1}{2}} \leq C\frac{1}{R}\left(\fint_{B_{\frac{3}{2}R}(X)} |u-c_0|^2\right)^{\frac{1}{2}} + C\left(\fint_{B_{\frac{3}{2}R}(X)} |\nabla\psi|^2\right)^{\frac{1}{2}}.$$

If $c_0 = \fint_{B_{2R}(X)} u$, the Poincaré inequality implies that there exists $1 < q < 2$ such that

$$\frac{1}{R}\left(\fint_{B_{2R}(X)} |u-c_0|^2\right)^{\frac{1}{2}} \leq C\left(\fint_{B_{2R}(X)} |\nabla u|^q\right)^{\frac{1}{q}}.$$

If $B_{\frac{3}{2}}(X) \cap \Omega^c \neq \emptyset$, then there exists a constant depending on the Lipschitz domain such that

$|\partial B_{2R}(X)| \le C|\partial B_{2R}(X) \cap \Omega^c|$. Thus if c_0 is defined as before, we have

$$\begin{aligned}\frac{1}{R}\left(\fint_{B_{2R}(X)} u^2\right)^{\frac{1}{2}} &\le \frac{1}{R}\left(\fint_{B_{2R}(X)} |u-c_0|^2\right)^{\frac{1}{2}} + \frac{1}{R}c_0 \\ &\le C\left(\fint_{B_{2R}(X)} |\nabla u|^q\right)^{\frac{1}{q}} + \frac{1}{R}\left(\fint_{B_{2R}(X)} u^q\right)^{\frac{1}{q}} \le C\left(\fint_{B_{2R}(X)} |\nabla u|^q\right)^{\frac{1}{q}}\end{aligned}$$

and so the Theorem is proven. □

Remark 2.4.3. *Lemma 2.4.2 can be extended to the case $divB = D$ in the weak sense, i.e. to $L \in \mathcal{O}$, where constants are weak solutions. The reason for that is because for fixed X and R, one can then assume that u is of the form $u - c_0$. This allows for the usage of the same proof.*

Lemma 2.4.4. *Using the setting of Lemma 2.4.2, we get*

$$||u||_{W_0^{1,p}(\Omega)} \le C||\psi||_{W^{1,p}(\Omega)}$$

for some $p > 2$.

Proof. Using Proposition 1.1, Chapter V, in [Gia83] (the proof is based on a Calderón–Zygmund decomposition), we get

$$||\nabla u||_{L^p(\Omega)} \le C||\nabla u||_{L^2(\Omega)} + C||\nabla\psi||_{L^p(\Omega)}.$$

Theorem 2.2.5 implies that $||u||_{W_0^{1,2}(\Omega)} \le C||\psi||_{W^{1,2}(\Omega)}$ and so the Cauchy–Schwarz inequality completes the proof. □

Lemma 2.4.5. *Let $L_j, L \in \mathcal{O}$ with $divB = D, divB_j = D_j$ and $||A_j - A||_{L^q(\Omega)} \to 0, ||\delta(\cdot)(B - B_j)||_{L^q((\partial\Omega)_\beta)} \to 0, ||B - B_j||_{L^q(\Omega_\beta)} \to 0$ for $1 < q < \infty$ large and the same sort of convergence for C and D as B. Let $\psi \in W^{1,2}(\Omega)$ and $u, u_j \in W_0^{1,2}(\Omega)$ be the solutions to $Lu = F_\psi = L_j u_j$, then $Lu_j \to Lu$ in $(W_0^{1,2}(\Omega))^*$.*

Proof. The proof is completed once we have shown that $\lim_j \sup_{\varphi\in W_0^{1,2}(\Omega)} \mathcal{L}(u-u_j,\varphi) = 0$. We have

$$\begin{aligned}\mathcal{L}(u_j,\varphi) &= \mathcal{L}_j(u_j,\varphi) + (\mathcal{L}-\mathcal{L}_j)(u_j,\varphi) \\ &= F_\psi(\varphi) + (\mathcal{L}-\mathcal{L}_j)(u_j,\varphi) \\ &= \mathcal{L}(u,\varphi) + (\mathcal{L}-\mathcal{L}_j)(u_j,\varphi).\end{aligned}$$

Thus it remains to show that $(\mathcal{L}-\mathcal{L}_j)(u_j,\varphi) \to 0$ uniformly in φ. This follows from Lemma 2.4.1 and the fact that $||u_j||_{W_0^{1,p}} \le C||\psi||_{W^{1,p}(\Omega)}$ for some $p > 2$ by Lemma 2.4.4. □

Lemma 2.4.6. *Using the setting from Lemma 2.4.5, we get $u_j \to u$ in $W_0^{1,2}(\Omega)$.*

Proof. We have seen that the Green's operator $\mathcal{G}$ is a bounded operator from $(W_0^{1,2}(\Omega))^*$ to $W_0^{1,2}(\Omega)$. This implies that

$$||u-u_j||_{W_0^{1,2}(\Omega)} = ||\mathcal{G}L(u-u_j)||_{W_0^{1,2}(\Omega)} \le C||L(u-u_j)||_{(W_0^{1,2}(\Omega))^*}.$$

Hence $||u-u_j||_{W_0^{1,2}(\Omega)} \to 0$ by Lemma 2.4.5. □

Lemma 2.4.7. *Let L, L_j, ψ be as in Lemma 2.4.5. Assume that v, v_j are the weak solutions to $Lv = L_j v_j = 0$ in Ω and $v = v_j = \psi$ on $\partial\Omega$. Then $v_j \to v$ in $W^{1,2}(\Omega)$.*

Proof. For u, u_j as in Lemma 2.4.5, we have $v = \psi - u$ and $v_j = \psi - u_j - w_j$ for $w_j \in W_0^{1,2}(\Omega)$ the solution to $L_j w_j = L_j\psi - L\psi$. Similarly to before, we have

$$\begin{aligned}||w_j||_{W_0^{1,2}(\Omega)} &= ||\mathcal{G}_j L_j w_j||_{W_0^{1,2}(\Omega)} \\ &\leq C||L_j w_j||_{(W_0^{1,2}(\Omega))^*} \leq C \sup_{||\varphi||_{W_0^{1,2}(\Omega)}=1} |\mathcal{L}_j - \mathcal{L}(\psi,\varphi)|.\end{aligned}$$

Since $\psi \in C^\infty(\mathbb{R}^n)$, Lemma 2.4.1 implies $||w_j||_{W_0^{1,2}(\Omega)} \to 0$ for $j \to \infty$ and then $||v - v_j||_{W^{1,2}(\Omega)} \leq ||u - u_j||_{W^{1,2}(\Omega)} + ||w_j||_{W^{1,2}(\Omega)} \to 0$ by Lemma 2.4.6. □

Corollary 2.4.8. *With the setting of Lemma 2.4.7 we get $v_j \to v$ uniformly in $\bar{\Omega}$.*

Proof. The remark after Lemma 2.2.10 establishes that v_j is equicontinuous, hence, by the Arzela-Ascoli theorem, a subsequence v_{j_k} converges uniformly to v.
Suppose $v_j \to v$ not uniformly in $\bar{\Omega}$. Then, for a $\varepsilon > 0$, there exists a subsequence v_{j_n} and points $X_n \in \Omega$ such that $|v_{j_n}(X_n) - v(X_n)| \geq \varepsilon$. Thus choosing k large enough such that $|v_{j_k} - v| \leq \frac{\varepsilon}{2}$, we have $|v_{j_n} - v_{j_k}| \geq \varepsilon/2$ for all n and all k large enough. Since v_j is equicontinuous, there exists a $\delta = \delta_\varepsilon$ such that $|v_j(X) - v_j(Y)| \leq \frac{1}{8}\varepsilon$ for all $j \geq 0$ and $Y \in B_\delta(X)$. Thus $|v_{j_n} - v_{j_k}| \geq \varepsilon/4$ on $B_\delta(X_n)$. This contradicts the fact that $v_j \to v$ in $L^2(\Omega)$. □

Corollary 2.4.9. *Let L_j, L be as in Lemma 2.4.5. Assume that $v_j, v \in W_{loc}^{1,2}(\Omega)$ are the weak solutions to $L_j v_j = Lv = 0$ in Ω and $v_j \equiv f \equiv v$ on $\partial\Omega$ for $f \in C^0(\partial\Omega)$. Then $v_j \to v$ uniformly and in $W_{loc}^{1,2}(\Omega)$.*

Proof. Let φ^k be smooth functions converging uniformly to f and let v_j^k, v^k be the weak solutions for L_j, L corresponding to the boundary data φ^k. Then the proof of Theorem 2.2.12 shows that v_j^k converges uniformly and in $W_{loc}^{1,2}(\Omega)$ to v_j as $k \to \infty$. The same applies to v^k and v. Since

$$v_j - v = (v_j - v_j^k) + (v_j^k - v^k) + (v^k - v)$$

and every term on the right converges uniformly and in $W_{loc}^{1,2}(\Omega)$, the Corollary is proven. □

Chapter 3

Green's Function

For a linear differential operator L, a function G is called the Green's function for L if $LG(X,Y) = \delta(X-Y)$ in the sense of distributions.
In [GW82], M. Grüter and K.O. Widman considered elliptic operators of the form $L_0 = \text{div}A\nabla$ on a bounded domain Ω, where A is as in the definition of $\mathcal{O}$. They prove that there exists a unique non-negative function $G : \Omega \times \Omega \to [0,\infty]$ such that for fixed $Y \in \Omega$ and every $r > 0$, the function G satisfies:

- $G(\cdot, Y) \in W^{1,2}(\Omega \backslash B_r(Y)) \cap W_0^{1,1}(\Omega)$
- $\mathcal{L}_0(G(\cdot,Y), \phi) = \phi(Y)$ for all $\phi \in C_0^\infty(\Omega)$.

In this chapter, we will show that the ideas in [GW82] can be extended to elliptic operators in[1] $\mathcal{O}_0$, i.e. we prove the existence of a Green's function for a certain class of elliptic operators with singular drift terms with no restriction on the smoothness of the matrix A.
If the coefficients of A are smooth, one sees that $G(\cdot, Y)$ is smooth in $\Omega\backslash\{Y\}$. Of that is taken advantage in [IR05] by A. Ifra and L. Riahi in order to prove that there exists a unique Green's function for elliptic operators with drift terms, which is comparable to the Green's function for the elliptic operator without drift term. Precisely, they considered operators of the form $L_1 u = L_0 u + B\nabla u$ for $L_0 = \text{div}A\nabla$ with A smooth and $B \in \mathcal{K}_{loc}(\Omega)$, where $\mathcal{K}_{loc}$ is defined as all measurable vector fields in the Kato class $K_{n+1}^{loc}(\Omega)$ (see Definitions 4.1 and 4.2 in [IR05]). The idea behind their proof is that the Green's function G_0 for operators of the form $L_0 = \text{div}A\nabla$ for smooth A are smooth away the diagonal and the gradient of the Green's function satisfies an estimate of the form

$$\frac{G_0(X,Z)|\nabla_Z G_0(Z,Y)|}{G_0(X,Y)} \leq C\left(\Gamma(X,Z) + \Gamma(Y,Z)\right),$$

for $\Gamma(X,Z) = \min\{1, \frac{\delta(Z)}{|X-Z|}\}\frac{1}{|X-Z|^{n-1}}$. This estimate allows one to express the Green's function for L_1 as an absolutely convergent infinite sum of terms involving G_0, see [IR05], Theorem 5.1.

3.1 The Existence of a Green's Function

We start this section by defining the weak L^p spaces

$$L_p^*(\Omega) = \{f : \Omega \to [0,\infty], f \text{ measurable and } ||f||_{L_p^*(\Omega)} < \infty\},$$

where $||f||_{L_p^*(\Omega)} = \sup_{t>0} t|\{f > t\}|^{\frac{1}{p}}$ and $\{f > t\} = \{x \in \Omega : f(x) > t\}$.

[1] The class $\mathcal{O}_0$ was defined as all elliptic operators in $\mathcal{O}$ with $C, D \equiv 0$ and B is divergence free in the sense of distributions, i.e. $L \in \mathcal{O}_0$ is of the form $Lu = \text{div}A\nabla u - B\nabla u$ for B a divergence free vector field.

Lemma 3.1.1. *For any measurable set Ω and $1 < p < \infty$, one has*

$$||f||_{L^p_*(\Omega)} \le ||f||_{L^p(\Omega)} \tag{3.1}$$

and

$$||f||_{L_{p-\varepsilon}(\Omega)} \le \left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}} |\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}} ||f||_{L^*_p(\Omega)}, \tag{3.2}$$

where $0 < \varepsilon \le p-1$.

Proof. The first inequality of the Lemma follows from Chebyshev's inequality:

$$\lambda|\{f > \lambda\}|^{\frac{1}{p}} \le \left(\int_{\{f(x)>\lambda\}} f(x)^p\right)^{\frac{1}{p}} \le ||f||_{L^p(\Omega)}.$$

For the second inequality, we use the fact that $||f||^q_{L^q(\Omega)} = q\int_0^\infty \lambda^{q-1}|\{f>\lambda\}|\mathrm{d}\lambda$ for $1 \le q < \infty$. Thus

$$||f||^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega)} = (p-\varepsilon)\left(\int_0^s \lambda^{p-\varepsilon-1}|\{f>\lambda\}|\,\mathrm{d}\lambda + \int_s^\infty \lambda^{p-\varepsilon-1}|\{f>\lambda\}|\,\mathrm{d}\lambda\right)$$

for any $s > 0$. Using $|\{f > \lambda\}| \le |\Omega|$ for the first term and the definition of the $||\cdot||_{L^p_*(\Omega)}$-norm for the second term, we get

$$||f||^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega)} \le s^{p-\varepsilon}|\Omega| - \frac{p-\varepsilon}{-\varepsilon}s^{-\varepsilon}||f||^p_{L^p_*(\Omega)}.$$

Choose $s = |\Omega|^{-\frac{1}{p}}||f||_{L^p_*(\Omega)}$, then

$$\begin{aligned}||f||^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega)} &\le |\Omega|^{-\frac{p-\varepsilon}{p}+1}||f||^{p-\varepsilon}_{L^p_*(\Omega)} + \frac{p-\varepsilon}{\varepsilon}|\Omega|^{\frac{\varepsilon}{p}}||f||^{-\varepsilon+p}_{L^p_*(\Omega)}\\ &= \frac{p}{\varepsilon}|\Omega|^{\frac{\varepsilon}{p}}||f||^{p-\varepsilon}_{L^p_*(\Omega)},\end{aligned}$$

which completes the proof. $\square$

The next theorem is the extension of (1.1) Theorem in [GW82] to elliptic operators $L \in \mathcal{O}_0$. The proof is an adjusted version of the proof in [GW82]. The assumption that B is divergence free in the sense of distributions will be used to prove a lower bound, which will be crucial in the proof.

Theorem 3.1.2. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. There exists a non-negative function $G : \Omega \times \Omega \to [0,\infty]$, such that for each $Y \in \Omega$ and any $R > 0$*

$$G(\cdot, Y) \in W^{1,2}(\Omega\backslash B_R(Y)) \cap W^{1,1}_0(\Omega) \tag{3.3}$$

and for all $\phi \in C_0^\infty(\Omega)$

$$\mathcal{L}(G(\cdot,Y),\phi) = \phi(Y), \tag{3.4}$$

i.e. $LG(X,Y) = -\delta_X(Y)$. We use the abbreviation $G(X) = G(X,Y)$ for $Y \in \Omega$. Then G

satisfies

$$G \in L^*_{\frac{n}{n-2}}(\Omega) \text{ with } ||G||_{L^*_{\frac{n}{n-2}}} \le C \tag{3.5}$$

$$\nabla G \in L^*_{\frac{n}{n-1}}(\Omega) \text{ with } ||\nabla G||_{L^*_{\frac{n}{n-1}}} \le C \tag{3.6}$$

$$G \in W_0^{1,s}(\Omega) \text{ for each } s \in [1, \frac{n}{n-1}). \tag{3.7}$$

$$G(X,Y) \le C|X-Y|^{2-n} \tag{3.8}$$

$$G(X,Y) \ge C|X-Y|^{2-n} \text{ for } X \in B_{\frac{1}{2}\delta(Y)}(Y). \tag{3.9}$$

where the constant C depends only on n, λ, ε_1, M and the Lipschitz domain Ω.

Proof. We indicate the step we are currently taking in the long proof in small boxes.

$\boxed{\text{Definition of } G_\rho\text{:}}$ in the proof of Theorem 2.2.6, we have seen that $\mathcal{L}$ is a bounded linear functional on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. In addition, for $u \in W_0^{1,2}(\Omega)$, we have $\int u\, B\nabla u = \frac{1}{2}\int B\nabla(u^2)$ and so $\mathcal{L}(u,u) = \int A\nabla u\ \nabla u$ since B is divergence free in the sense of distributions. Thus the bounded, bilinear functional $\mathcal{L}$ is coercive. Fix $\rho > 0$. For $B_\rho = B_\rho(Y)$, we define on $W_0^{1,2}(\Omega)$ the bounded, linear functional

$$\phi \mapsto \fint_{B_\rho(Y)} \phi.$$

Theorem 2.2.5 (or one can use the Lax Milgram Theorem directly) implies the existence of $G_\rho \in W_0^{1,2}(\Omega)$ such that $\mathcal{L}(G_\rho, \phi) = \fint_{B_\rho} \phi$ for all $\phi \in W_0^{1,2}(\Omega)$.

$\boxed{G_\rho \text{ is non-negative:}}$ for $u \in W_0^{1,2}(\Omega)$, one obviously has $\mathcal{L}(|u|,|u|) = \mathcal{L}(u,u)$ and $\mathcal{L}(|u|,u) = \mathcal{L}(u,|u|)$. Inserting G_ρ as a test-function, we see that $0 \le \mathcal{L}(G_\rho, G_\rho) = \fint_{B_\rho} G_\rho = K \fint_{B_\rho} |G_\rho| = K\mathcal{L}(G_\rho, |G_\rho|)$ for some $K \ge 1$. This implies

$$\mathcal{L}(G_\rho, G_\rho) = \mathcal{L}\left(G_\rho, \frac{|G_\rho|}{K}\right) = \mathcal{L}\left(\frac{|G_\rho|}{K}, G_\rho\right),$$
$$\mathcal{L}\left(\frac{|G_\rho|}{K}, \frac{|G_\rho|}{K}\right) = \frac{1}{K^2}\mathcal{L}\left(G_\rho, G_\rho\right) \le \mathcal{L}\left(G_\rho, \frac{|G_\rho|}{K}\right).$$

Hence $\mathcal{L}(K^{-1}|G_\rho| - G_\rho, K^{-1}|G_\rho| - G_\rho) \le 0$, i.e. $G_\rho - \frac{|G_\rho|}{K} = 0$, and so $G_\rho \ge 0$.

$\boxed{||G_\rho||_{L^*_{\frac{n}{n-2}}(\Omega)} \le C\text{:}}$ we use $\phi = [\frac{1}{t} - \frac{1}{G_\rho}]_+$ as a test function. First observe that for $\Gamma_t = \{G_\rho > t\}$ one has, by the assumption that B is divergence free,

$$-\int_\Omega B\nabla G_\rho\, \phi = \int_{\Gamma_t} G_\rho\, B\frac{\nabla G_\rho}{G_\rho^2}$$
$$= \int_{\Gamma_t} B\frac{\nabla G_\rho}{G_\rho} = \int_\Omega B\nabla([\ln G_\rho - \ln t]_+) = 0.$$

Thus the zero divergence of B implies

$$C_\lambda \int_{\Gamma_t} \left(\frac{|\nabla G_\rho|}{G_\rho}\right)^2 \le \int_{\Gamma_t} A\nabla G_\rho \cdot \frac{\nabla G_\rho}{G_\rho^2} = \mathcal{L}(G_\rho, \phi) = \fint_{B_\rho} \phi \le \frac{1}{t}.$$

Using Sobolev's inequality for $[\ln G_\rho - \ln t]_+$, we get

$$\left(\int_{\Gamma_t} \left[\ln\left(\frac{G_\rho}{t}\right)\right]^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C\frac{1}{t}.$$

The fact that $\ln(\frac{G_\rho}{t}) \ge C$ on Γ_{2t} implies $t|\{G_\rho > t\}|^{\frac{n-2}{n}} \le C$ and therefore $||G_\rho||_{L^*_{\frac{n}{n-2}}(\Omega)} \le C$.

$\boxed{\int_\Omega |\nabla G_\rho|^2 \le C\rho^{2-n}:}$ since $\mathcal{L}$ is coercive, the usage of G_ρ as a test function gives us

$$\int_\Omega |\nabla G_\rho|^2 \le C \fint_{B_\rho} G_\rho \le \left(\fint_{B_\rho} G_\rho^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \le C\rho^{\frac{2-n}{2}} \left(\int_\Omega |\nabla G_\rho|^2 \right)^{\frac{1}{2}},$$

where we use the Poincaré inequality on Ω for the last inequality.

$\boxed{G_\rho(X) \le C|X-Y|^{2-n} \text{ for } |X-Y| \ge 2\rho:}$ let $R = |X-Y| \ge 2\rho$. If $\delta(X) \ge \frac{R}{10}$, then G_ρ is a solution on $B_{\frac{R}{10}}(X)$ and so Harnack's inequality and (3.2) with $p = n/(n-2)$ and $\varepsilon = p-1$ (therefore $\frac{\varepsilon}{p(p-\varepsilon)} = \frac{2}{n}$) imply

$$G_\rho(X) \le C \fint_{B_{\frac{R}{20}}(X)} G_\rho \le C\frac{1}{R^n} |B_R(X)|^{\frac{2}{n}} ||G_\rho||_{L^*_{\frac{n}{n-2}}(\Omega)}.$$

Hence $G_\rho(X) \le CR^{2-n}$ for $\delta(X) \ge \frac{R}{10}$. If $\delta(X) \le \frac{R}{10}$, then $B_{\frac{R}{10}}(X) \cap B^\rho = \emptyset$ and so G_ρ is a solution on $T_{\frac{1}{10}R}(\hat{X})$. Hölder continuity up to the boundary gives us

$$G_\rho(X) \le C \fint_{T_{\frac{R}{10}}(\hat{X})} G_\rho.$$

Hence (3.2) implies the same bound as before.

$\boxed{||\nabla G_\rho||_{L^*_{\frac{n}{n-1}}(\Omega)} \le C:}$ fix $R > 0$ and assume first that $R > 4\rho$. Let $\eta \equiv 1$ outside of $B_R(Y)$ and $\eta \equiv 0$ in $B_{\frac{R}{2}}$ with $|\nabla \eta| < C/R$. Using $G_\rho \eta^2$ as a test function, we get $\mathcal{L}(G_\rho, G_\rho\eta^2) = 0$, i.e.

$$\int_\Omega A\nabla G_\rho \cdot \nabla G_\rho \, \eta^2 + 2\int_\Omega A\nabla G_\rho \cdot \nabla \eta \, G_\rho \eta + \int_\Omega B\nabla G_\rho \, G_\rho \eta^2 = 0.$$

By Lemma 2.1.2, we have

$$|\int_\Omega B\nabla G_\rho \, G_\rho \eta^2| \le (\varepsilon + C_\varepsilon \varepsilon_1) \int_\Omega |\nabla G_\rho| \eta^2 + C_\varepsilon \int_\Omega (\eta^2 + |\nabla \eta|^2) G_\rho^2.$$

Thus, by ellipticity and by the pointwise estimate $G_\rho(X) \le C|X-Y|^{2-n}$ for $|X-Y| > 2\rho$ we get

$$\int_{\Omega \backslash B_R(Y)} |\nabla G_\rho|^2 \eta^2 \le (C_\varepsilon + C_\varepsilon \varepsilon_1) \int_\Omega (\eta^2 + |\nabla \eta|^2) G_\rho^2 \le CR^{2-n}.$$

For the $R < 4\rho$ case, we use $\int_\Omega |\nabla G_\rho|^2 \le C\rho^{2-n} \le CR^{2-n}$. For any $R > 0$ we get

$$\int_{\Omega \backslash B_R(Y)} |\nabla G_\rho|^2 \le CR^{2-n}.$$

Define $\tilde{\Gamma}_t = \{|\nabla G_\rho| > t\}$ and $R = t^{-\frac{1}{n-1}}$, then (3.1) implies $t^2|\{\tilde{\Gamma}_t \cap \Omega \backslash B_R(Y)\}| \le Ct^{\frac{n-2}{n-1}}$. Trivially, $t^2|B_{R(Y)}| \le Ct^2 t^{-\frac{n}{n-1}} = Ct^{\frac{n-2}{n-1}}$, and so $t^{\frac{n}{n-1}}|\tilde{\Gamma}_t| \le C$.

$\boxed{\text{Definition of } G:}$ We have seen that for each $s \in [1, \frac{n}{n-1}), G_\rho$ is uniformly in $W_0^{1,s}(\Omega)$ with respect to ρ. By considering sequences $\rho_\nu \searrow 0$ and $s_\nu \nearrow \frac{n}{n-1}$, we can find a subsequence $\{G_{\rho_\mu}\}$ of the sequence $\{G_{\rho_\nu}\}$ and a function $G \in W_0^{1,s}(\Omega)$ for all $s < \frac{n}{n-1}$ such that $G_{\rho_\mu} \rightharpoonup G$ weakly in $W_0^{1,s}(\Omega)$ for $s \in [1, \frac{n}{n-1})$. Hence, (3.7) is satisfied.
In the proof of Theorem 2.2.3, we have seen that for fixed $\phi \in C_0^\infty(\Omega)$, $\mathcal{L}(\cdot, \phi)$ is a bounded

linear functional on $W_0^{1,s}(\Omega)$. Hence $\mathcal{L}(G_{\rho_\mu},\phi) \to \mathcal{L}(G,\phi)$ and

$$\mathcal{L}(G,\phi) = \lim_{\rho_\mu} \mathcal{L}(G_{\rho_\mu},\phi) = \lim_{\rho_\mu} \fint_{B_{\rho_\mu}} \phi = \phi(Y).$$

L^p norms are weakly lower-semicontinuous[2]. So for $\Gamma_t = \{G > t\}, p = \frac{n}{n-2}$ and $0 < \varepsilon < p-1$, we get

$$\begin{aligned} t^{p-\varepsilon}|\Gamma_t| \le ||G||^{p-\varepsilon}_{L^{p-\varepsilon}(\Gamma_t)} &\le \lim_{\mu\to\infty} \inf ||G_{\rho_\mu}||^{p-\varepsilon}_{L^{p-\varepsilon}(\Gamma_t)} \\ &\le \lim_{\mu\to\infty} \inf \left(\frac{p}{\varepsilon}\right) |\Gamma_t|^{\frac{\varepsilon}{p}} ||G_{\rho_\mu}||^{p-\varepsilon}_{L^p_*(\Gamma_t)}. \end{aligned}$$

Thus $t|\Gamma_t|^{\frac{1}{p}} \le C\left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$. For $\varepsilon \to p-1$, we get $||G||_{L^*_{\frac{n}{n-2}}} \le C$, which is (3.5). The same argument gives us $||\nabla G||_{L^*_{\frac{n}{n-1}}} \le C$, i.e. (3.6) holds.
We have seen that $\int_{\Omega\backslash B_R} |\nabla G_\rho|^2 \le CR^{2-n}$ for $R > 4\rho$, hence by going to another subsequence we can assume that $G_{\rho_\mu} \rightharpoonup G$ in $W_0^{1,2}(\Omega\backslash B_R(Y))$ and $\int_{\Omega\backslash B_R(Y)} |\nabla G|^2 \le CR^{2-n}$. Using the Rellich–Kondrachov compactness Theorem (e.g. see [Eva98], page 272), we see that $G_{\rho_\mu} \to G$ in $L^1(\Omega)$. Thus by continuity $G_{\rho_\mu} \to G$ everywhere in $\Omega\backslash\{Y\}$ and so $G(X) \le C|X-Y|^{2-n}$. This means that (3.8) is true.
We are not going to prove uniqueness. So let G be any function satisfying (3.3) and (3.4). Fix $X, Y \in \Omega$ and assume that $R = |X-Y| \le \frac{1}{2}\delta(Y)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be 1 on $B_{\frac{1}{2}R}(Y)$ and 0 outside of $B_R(Y)$ with $|\nabla\phi| \le \frac{C}{R}$, then

$$\begin{aligned} 1 = \mathcal{L}(G,\phi) &= \int A\nabla G \cdot \nabla\phi + B\nabla G\,\phi \\ &\le \frac{C}{R}\int_{B_R(Y)\backslash B_{R/2}(Y)} |\nabla G| + \frac{C}{R^2}\int_{B_R(Y)\backslash B_{R/2}(Y)} G, \end{aligned}$$

where we have used the fact that B is divergence free and that $R \le \frac{1}{2}\delta(Y)$ and so $|B(X)| \le C/R$ on $B_R(Y)\backslash B_{R/2}(Y)$. The Cacciopoli inequality and the Harnack principle imply

$$1 \le \frac{C}{R}R^n\left(\fint_{B_R(Y)\backslash B_{R/2}(Y)} |\nabla G|^2\right)^{\frac{1}{2}} + G(X,Y)R^{n-2} \le CR^{n-2}G(X,Y),$$

which is (3.9). Thus all statements of the Theorem are proven. □

Since we did not prove uniqueness, we will define THE Green's function as the Green's function constructed in the previous Theorem.
For the application later on, we need a result about the adjoint. Let G^* denote the Green's function for the adjoint $L^*v = \mathrm{div}A^T\nabla v - B\nabla v$, where A^T is the transpose of A.

Lemma 3.1.3. *Let a_{jk} be a double sequence in $\mathbb{R}$ with $\lim_j \lim_k a_{jk} = a, \lim_k \lim_j a_{jk} = b$. Assume that $\lim_j a_{jk} = a_{\infty k}$ uniformly in k and $\lim_k a_{jk} = a_{j\infty}$ uniformly in j. Then $a = b$.*

Proof. We have

$$|a-b| \le |a - a_{j\infty}| + |a_{j\infty} - a_{jk}| + |a_{jk} - a_{\infty k}| + |a_{\infty k} - b|.$$

The assumptions imply:

- there exists $N_1 = N_1(\varepsilon)$ such that $|a_{jk} - a_{\infty k}| < \varepsilon$ for all $j > N_1$.
- there exists $N_2 = N_2(\varepsilon)$ such that $|a - a_{j\infty}| < \varepsilon$ for all $j > N_2$.

[2] Let X be a Banach space, X^* its dual and assume $X \ni x_n \rightharpoonup x$ weakly in X. Then, for every $x^* \in X^*$ with $||x^*||_{X^*} = 1$, we have $x^*(x) = \liminf x^*(x_n) \le \liminf ||x_n||$. Since $||x||_X = \sup_{||x^*||_{X^*}=1} x^*(x)$, it follows that the norm is weakly lower-semicontinuous, i.e. $||x||_X \le \liminf ||x_n||_X$.

- there exists $N_3 = N_3(\varepsilon)$ such that $|a_{jk} - a_{j\infty}| < \varepsilon$ for all $k > N_3$.
- there exists $N_4 = N_4(\varepsilon)$ such that $|b - a_{\infty k}| < \varepsilon$ for all $k > N_4$.

Thus for $j, k > \max\{N_1, N_2, N_3, N_4\}$ we have $|a - b| < 4\varepsilon$. $\square$

Theorem 3.1.4. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Then the Green's functions G and G^* for L and L^* with $L \in \mathcal{O}_0$ satisfy $G^*(Y,X) = G(X,Y)$ for $X \neq Y$.*

Proof. The proof is taken from the proof (1.3) Theorem in [GW82]. For $X \neq Y$, we have sequences $\{\rho_\nu\}$ and $\{\sigma_\mu\}$ tending towards zero with $\rho_\nu, \sigma_\mu < \frac{1}{3}|X - Y|$ such that $G_{\rho_\nu}(\cdot, Y) \to G(\cdot, Y)$ and $G^*_{\sigma_\mu}(\cdot, X) \to G^*(\cdot, X)$ almost everywhere. Using them as test functions and the fact that L^* is the adjoint of L results in

$$\fint_{B_{\rho_\nu}(Y)} G^*_{\sigma_\mu}(\cdot, X) = \fint_{B_{\sigma_\mu}(X)} G_{\rho_\nu}(\cdot, Y) =: a_{\nu\mu}.$$

Since $G_{\rho_\nu}(\cdot, Y)$ is continuous on $B_{\sigma_\mu}(X)$ we get

$$\fint_{B_{\rho_\nu}(Y)} G^*(\cdot, X) = G_{\rho_\nu}(X, Y)$$

by sending $\sigma_\mu \to 0$. Sending ρ_ν to zero then implies $G^*(Y,X) = \lim_{\rho_\nu \to 0} G_{\rho_\nu}(X,Y)$. Since $G^*(\cdot, X)$ is Hölder continuous away from X, the convergence is uniform. Repeating these steps with G and G^* replaced gives, by Lemma 3.1.3,

$$G^*(Y,X) = G(X,Y), \qquad X \neq Y,$$

which completes the proof. $\square$

For a further application later in this thesis, we will need (1.8) Theorem from [GW82], which we will prove with a slightly different proof than in [GW82].

Lemma 3.1.5. *Let $L \in \mathcal{O}_0$, G be the corresponding Green's function and Ω be a Lipschitz domain. Then, there exists $\alpha > 0$ such that*

$$G(X,Y) \leq C\delta(Y)^\alpha |X - Y|^{2-n-\alpha}.$$

Proof. As in [GW82], we divide the proof into several cases.
$\boxed{\delta(Y) > \frac{1}{10}|X - Y|:}$ by (3.8), we get

$$G(X,Y) \leq C|X - Y|^{2-n} \leq C\delta(Y)^\alpha |X - Y|^{2-n-\alpha}$$

for every $\alpha > 0$.
$\boxed{\delta(Y), \frac{1}{10}|X - Y| > \frac{1}{10}R_0:}$ obvious.
$\boxed{\delta(Y) \leq \frac{1}{10}R_0 \leq \frac{1}{10}|X - Y|:}$ the conditions imply, $X \notin B_{\frac{1}{2}R_0}(Y)$. Therefore, we can use the Hölder continuity up to the boundary on $T_{\frac{1}{4}R_0}(\hat{Y}) \subset B_{\frac{1}{2}R_0}(Y)$ and the boundedness of Ω to get

$$G(X,Y) \leq C\left(\frac{\delta(Y)}{R_0}\right)^\alpha G(X, A_{\frac{1}{10}R_0}(\hat{Y})) \leq C\delta(Y)^\alpha |X - Y|^{2-n-\alpha}.$$

$\boxed{\delta(Y) \leq \frac{1}{10}|X - Y| \leq \frac{1}{10}R_0:}$ define $\Sigma = T_{\frac{5}{10}|X-Y|}(\hat{Y})$. Let u be the weak solution for L^* in Σ corresponding to the following continuous boundary data on $\partial\Sigma$:

$$\begin{cases} \equiv 1 & \text{on } \partial\Sigma \cap \Omega \\ \equiv 0 & \text{on } \Delta_{\frac{4}{10}|X-Y|}(\hat{Y}) \\ \in [0,1] & \text{otherwise.} \end{cases}$$

For any $Z \in \partial\Sigma \cap \Omega$ we have $|X - Z| \geq C|X - Y|$ and therefore, $G(X, Z) \leq C|X - Y|^{2-n}$. Hence the maximum principle implies

$$G(X, Z) \leq C|X - Y|^{2-n} u(Z)$$

for all $Z \in \Sigma$. By the Hölder continuity up to the boundary, we get $u(Z) \leq C\left(\frac{\delta(Z)}{|X-Y|}\right)^{\alpha}$ for any $Z \in T_{\frac{2}{10}|X-Y|}(\hat{Y})$. Thus, $G(X, Y) \leq C|X - Y|^{2-n-\alpha}\delta(Y)^{\alpha}$, which completes the proof. □

3.2 A Representation of the Green's operator based on the Green's Function

Having proven the existence of a Green's function for $L \in \mathcal{O}_0$ with Ω a Lipschitz domain, we will now show how the Green's function is related to the Green's operator. We have seen that the following two maps are bounded:

- the map $L : W^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ defined by $u \mapsto Lu = \mathcal{L}(u, \cdot)$.
- the Green's operator $\mathcal{G} : (W_0^{1,2}(\Omega))^* \to W_0^{1,2}(\Omega)$ with $\mathcal{G}(F)$ defined as the unique solution in $W_0^{1,2}(\Omega)$ to $L\mathcal{G}(F) = F$.

This implies that L, seen as an operator from $W_0^{1,2}(\Omega)$ to $(W_0^{1,2}(\Omega))^*$, is surjective.
For $\psi \in C_0^{\infty}(\Omega)$, we define

$$G(\psi)(X) = \int_{\Omega} G(X, Y)\psi(Y)\,\mathrm{d}Y.$$

We claim that $G(\psi)$ is in $W_0^{1,2}(\Omega)$ with $||G(\psi)||_{W_0^{1,2}(\Omega)} \leq C||\psi||_{(W_0^{1,2}(\Omega))^*}$. The boundedness of $\mathcal{G}$ and L implies

$$||\phi||_{W_0^{1,2}(\Omega)} = ||\mathcal{G}L\phi||_{W_0^{1,2}(\Omega)} \leq C||L\phi||_{(W_0^{1,2}(\Omega))^*} \leq C||\phi||_{W_0^{1,2}(\Omega)}.$$

Using the fact that for $z \in Z$ with Z a Banach space, $||z||_Z = \sup_{||z^*||_{Z^*}=1} z^*(z)$, Lemma A.0.28 in the Appendix and the fact that L is surjective, the statement $||G(\psi)||_{W_0^{1,2}(\Omega)} \leq C||\psi||_{(W_0^{1,2}(\Omega))^*}$ is equivalent to

$$|\mathcal{L}^*(\phi, G(\psi))| \leq C||\phi||_{W_0^{1,2}(\Omega)}||\psi||_{(W_0^{1,2}(\Omega))^*} \tag{3.10}$$

for all $\phi \in C_0^{\infty}(\Omega)$, where $L^*u = \mathrm{div}(A^T\nabla u + Bu)$. Integration by parts gives us

$$\mathcal{L}^*(\phi, G(\psi)) = \int_{\Omega} \mathcal{L}(G(\cdot, Y), \phi)\psi(Y)\,\mathrm{d}Y = \int_{\Omega} \phi(Y)\psi(Y)\,\mathrm{d}Y.$$

Thus (3.10) holds and so $G(\psi) \in W_0^{1,2}(\Omega)$ with $LG(\psi) = \psi$. This means that $G(\psi)$ is a solution in $W_0^{1,2}(\Omega)$ to $LG(\psi) = \psi$. This solution is uniquely given by $\mathcal{G}(\psi)$, hence $G(\psi) = \mathcal{G}(\psi)$. The inequality (3.10) allows the extension to $(W_0^{1,2}(\Omega))^*$ and so we conclude that $G(\psi) = \mathcal{G}(\psi)$ for all $\psi \in (W_0^{1,2}(\Omega))^*$.

Chapter 4

The Elliptic Measure

In this chapter, we will introduce the elliptic measure for elliptic operators $L \in \mathcal{O}$, which is a consequence of the solvability of the continuous Dirichlet problem and the maximum principle. Furthermore, we will follow the ideas in [CFMS81] to prove that the elliptic measure for $L \in \mathcal{O}_0$ is doubling and that weak solutions to $L \in \mathcal{O}_0$, which vanish at a part of the boundary, satisfy a comparison principle.

4.1 Definition of the Elliptic Measure

In order to introduce the elliptic measure, we will need the following definitions from [Rud66], 2.15 Definition and 2.16 Definition:

Definition 4.1.1. *A non-negative Borel measure μ on a locally compact Hausdorff space X is called regular if and only if every Borel set E is outer and inner regular, i.e.*

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\},$$
$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\},$$

where the second equality has to hold if E is open or E is Borel with $\mu(E) < \infty$.
A set E is called σ-compact if E is a countable union of compact sets.

Theorem 4.1.1 (Riesz Representation Theorem)**.** *Let X be a locally compact, σ-compact Hausdorff space. For any positive linear functional Λ on $C_0^0(X)$, there exists a unique regular Borel measure μ such that for all $g \in C_0^0(X)$ one has*

$$\Lambda(g) = \int_X g \, d\mu.$$

The regularity of μ implies that for every Borel set E with $\mu(E) < \infty$ and every $\varepsilon > 0$, there exist a closed set B, with $B \subset E$ and an open set A with $E \subset A$, both depending on $\varepsilon > 0$, being such that $\mu(A \backslash B) < \varepsilon$.

Proof. See [Rud66] 2.14 Theorem and 2.17 Theorem. □

In Theorem 2.2.12, we have seen that if Ω is a Lipschitz domain and $L \in \mathcal{O}$, there exists a unique $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ for every $g \in C^0(\partial\Omega)$, such that $Lu = 0$ in Ω and $u = g$ on $\partial\Omega$. By the maximum principle, we have $||u||_{L^\infty(\Omega)} \leq ||g||_{L^\infty(\Omega)}$. Thus for every fixed $X \in \Omega$ the map defined by

$$C^0(\partial\Omega) \ni g \mapsto u(X)$$

is a positive, bounded linear functional on $C^0(\partial\Omega)$. The Riesz Representation Theorem implies the existence of a unique regular Borel measure w_L^X such that

$$u(X) = \int_{\partial\Omega} g(Q) \, \mathrm{d}w_L^X(Q),$$

i.e. every $L \in \mathcal{O}$ gives rise to a family of regular Borel measures $\{w_L^X\}_{X\in\Omega}$. If it is clear about which family of regular measures we are speaking, we will omit the index L and write w^X instead of w_L^X. Moreover, we will write w instead of w^{X_0} if we speak about a fixed X_0. Without losing generality, we can assume that $0 \in \Omega$ and that this fixed $X_0 = 0$. So w is to be understood as w^0.
Let $\mathcal{B}$ be the collection of Borel sets (i.e. the smallest σ-algebra containing all open subsets of $\partial\Omega$) and define

$$\mathcal{N}_X = \{N \subset B \in \mathcal{B} : w^X(B) = 0\},$$
$$\mathcal{W}_X = \{N \cup B : N \in \mathcal{N}_X, B \in \mathcal{B}\}.$$

Then w^X can be extended to a measure on $\mathcal{W}_X$. The sets in $\mathcal{W}_X$ are the w^X-measurable sets. The following Lemma shows that $\mathcal{W}_X$ is independent of X.

Lemma 4.1.2. *Let $L \in \mathcal{O}$ and $E \in \mathcal{B}$ with $w^{X_0}(E) = 0$ for some $X_0 \in \Omega$. Then $w^X(E) = 0$ for all $X \in \Omega$.*

Proof. We follow the proof [Ken94], Lemma 1.2.7. We know that w^X is a regular Borel measure. Thus, by the Definition 4.1.1, it is enough to show that $w^{X_0}(K) = 0$ for any compact K and some $X_0 \in \Omega$ implies $w^X(K) = 0$ for all $X \in \Omega$.
Regularity of w^{X_0} says that for $\varepsilon > 0$, there exists an open $U = U_\varepsilon \subset \partial\Omega$ such that $K \subset U$ and $w^{X_0}(U) \le \varepsilon$. Urysohn's Lemma (see [Rud66], page 39) provides the existence of $g \in C^0(\partial\Omega)$ with $0 \le g \le 1, g \equiv 1$ on K and $\operatorname{supp} g \subset U$. Let v be the weak solution for L with boundary values g. Harnack's inequality implies that for $X \in \Omega$ there exists $C_{(X,X_0)}$ such that

$$v(X) \le C_{(X,X_0)} v(X_0) \le C_{(X,X_0)}\varepsilon,$$

i.e. $w^X(E) \le C_{(X,X_0)}\varepsilon$. Since the choice of U is independent of X and X_0, the proof is complete. $\square$

Lemma 4.1.2 tells us that the family of regular measures $\{w^X\}_{X\in\Omega}$ is absolutely continuous with respect to each other. Thus, by the Radon-Nikodym Theorem (e.g. see [Rud66], 6.9 Theorem) there exists a unique $K(X, \cdot) \in L^1(\partial\Omega, \mathrm{d}w^{X_0})$ for a fixed $X_0 \in \Omega$ such that

$$K(X, Q) = \frac{\mathrm{d}w^X}{\mathrm{d}w^{X_0}}(Q). \tag{4.1}$$

Definition 4.1.2. *For $X_0 \in \Omega$, we define the σ-algebra $\mathcal{W} = \mathcal{N}_{X_0} \cup \mathcal{B}$ (which is well-defined by Lemma 4.1.2). The family $\{w_L^X\}_{X\in\Omega}$ of regular Borel measures on the σ-algebra $\mathcal{W}$ is called the family of elliptic measures corresponding to L. For a fixed $X_0 \in \Omega$, we call w^{X_0} the elliptic measure for L at X_0. Sets in $\mathcal{W}$ are called w-measurable.*

More details about the elliptic measure (which is called the harmonic measure for $L = \Delta$ in the literature) can be found in [Hel75] for example.
An application of the definition of the elliptic measure is the following representation formula for $L \in \mathcal{O}_0$, for which we will need the existence of a Green's function: let $\phi \in C^\infty(\mathbb{R}^n)$. Then $u(X) = \int_{\partial\Omega} \phi(Q)\, \mathrm{d}w^X(Q)$ is the weak solution with boundary data ϕ. Moreover, $\phi - u \in W_0^{1,2}(\Omega)$ and $\phi - u$ satisfies $L(\phi - u) = L\phi$. Thus, $\phi - u = \mathcal{G}(L\phi)$ and so integration by parts gives us

$$\phi(X) = \int_{\partial\Omega} \phi\, \mathrm{d}w^X + \int_\Omega A\nabla\phi(Y) \cdot \nabla_Y G(X, Y) + G(X, Y)\, B\nabla\phi(Y)\, \mathrm{d}Y. \tag{4.2}$$

Having defined the elliptic measure, it is natural to ask if the elliptic measure is absolutely continuous with respect to the surface measure. In general, the answer is false. In [CFK81] L. Caffarelli, E. Fabes and C.E. Kenig have constructed an elliptic operator $L_0 = \mathrm{div}A\nabla$, for which the elliptic measure w is completely singular with respect to the surface measure. The proof is based on a Theorem of Beurling and Ahlfors in [BA56] on quasi-conformal maps.
If one has an elliptic operator for which the previous question is answered with yes, one might

wonder how "nicely" the Radon-Nikodym derivative $\frac{\mathrm{d}w}{\mathrm{d}\sigma}$, called the Poisson kernel, behaves. We will shed some light on this question later on. Once we have defined what is meant by solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$, we will see that a reverse Hölder property of the Poisson kernel is equivalent to the solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$.

4.2 The Doubling Property and a Comparison Theorem for $L \in \mathcal{O}_0$

In this section, we use the ideas in [CFMS81] to prove that the elliptic measure for $L \in \mathcal{O}_0$ is a doubling measure. This result will allow us to prove a comparison Theorem for weak solutions to $L \in \mathcal{O}_0$ as in [CFMS81].
Firstly, we will show that $w^X(E)$ for $E \in \mathcal{W}$ and $L \in \mathcal{O}$ is a weak solution. For this, we need the following Lemma:

Lemma 4.2.1. *Let $L \in \mathcal{O}$ and Ω be a Lipschitz domain. Assume $u_j \in W^{1,2}_{loc}(\Omega)$ are weak solutions which converge on all compact subsets $K \subset \Omega$ uniformly to a function u, then $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution.*

Proof. Fix $K \subset \Omega$. Then by the assumption on $\{u_j\}_j$, there exists $N_{\varepsilon,K}$ such that $||u_j - u_k||_{L^\infty(K)} \leq \varepsilon$ for all $k, j \geq N_{\varepsilon,K}$. Let $K' \subset O \subset K \subset \Omega$ for K' compact and O open. Then Caccioploi's inequality, applied to the non-negative functions $[u_j - u_k]_+$ and $[u_k - u_j]_+$, which are subsolutions by Theorem 2.2.14, implies that

$$\int_{K'} |\nabla(u_j - u_k)|^2 \leq C_{(K,K')}\varepsilon$$

for $j, k \geq N_{\varepsilon,K}$. Thus $\{\nabla u_j\}_j$ is a Cauchy sequence in $L^2(K')$ and converges to a vector field $F = (F_1, \cdots, F_n) \in L^2(K')$. By the definition of weak derivatives, we see that for $\phi \in C_0^\infty(K')$, one has

$$\int_{K'} F_i\phi = \lim_j \int_{K'} \partial_i u_j \phi = -\lim_j \int_{K'} u_j \partial_i \phi = -\int_{K'} u\partial_i\phi,$$

i.e. $F = \nabla u$ on K'. The coefficients of L are in $L^\infty(K')$ hence

$$\mathcal{L}(u, \phi) = \lim_j \mathcal{L}(u - u_j, \phi) = 0$$

for all $\phi \in C_0^\infty(K')$. Since for every $\phi \in C_0^\infty(\Omega)$ one can find $K' \subset O \subset K \subset \Omega$ as above with $\operatorname{supp}\phi \subset K'$, we see that u is a weak solution with $u \in W^{1,2}_{loc}(\Omega)$. □

Lemma 4.2.2. *Let $L \in \mathcal{O}$ and Ω be a Lipschitz domain. For $E \in \mathcal{W}$ we have that $w^X(E)$ is a weak solution.*

Proof. The proof can be found in [Ken94], whereas we will add some details. First, we show that the conclusion of the Lemma holds for open sets $U \in \mathcal{B}$ and second, for general sets in $\mathcal{W}$. Choose $K_j \subset U$ compact with $w^X(U\backslash K_j) \to 0$ and g_j for K_j and u as g for K and u in the proof of Lemma 4.1.2. We call the corresponding solutions v_j. In the proof of Lemma 4.1.2, we have seen that v_j converges uniformly on any compact subset to $w^X(U)$. Thus, $w^X(U)$ is a weak solution by Lemma 4.2.1.
Having proven the Lemma for open sets, we consider general $E \in \mathcal{W}$. For $E \in \mathcal{W}$ we can find, on the one hand, open sets U_j with $U_j \downarrow$ and a set $Z \in \mathcal{W}$ with $w(Z) = 0$ such that $E = \bigcap_j U_j \cup Z$. On the other hand, let $K_j \subset E$ be compact with $K_j \uparrow, \bigcup_j K_j \cup \bar{Z} = E$ for another w-null set $\bar{Z}$. Then the regularity of the elliptic measure implies

$$0 \leq w^{X_0}(U_j) - w^{X_0}(E) \leq w^{X_0}(U_j\backslash K_j) \leq \varepsilon$$

for j large enough.
Since $U_j\backslash K_j$ is open, we can apply Harnack's inequality to get $w^X(U_j\backslash K_j) \leq C_K\varepsilon$ for all X

in a fixed compact K. Thus $w^X(U_j) \to w^X(E)$ uniformly in X on K. Lemma 4.2.1 completes the proof. $\square$

We are now able to prove that the elliptic measure $w_L^{X_0}$ for $L \in \mathcal{O}_0$ is doubling. We adapt the proof in [CFMS81] to elliptic operators in $\mathcal{O}_0$.

Lemma 4.2.3. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}_0$ with G and w^X the corresponding Green's function and elliptic measure. Choose $Q_0 \in \partial\Omega$ and $R \leq R_0$, then for $X \in \Omega \backslash T_{4R}(Q_0)$ one has*

$$\frac{1}{C} R^{n-2} G(X, A_R(Q_0)) \leq w^X(\Delta_R(Q_0)) \leq C R^{n-2} G(X, A_R(Q_0)).$$

Proof (compare [CFMS81]). Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \varphi \leq 1, \varphi \equiv 1$ on $B_R(Q_0), \varphi \equiv 0$ on $B_{2R}(Q_0)^c$ and $|\nabla\varphi| \leq \frac{C}{R}$. Then, applying the representation (4.2) for the weak solution u corresponding to $Lu = 0$ in Ω and $u \equiv \varphi$ on $\partial\Omega$ gives us

$$\begin{aligned}
w^X(\Delta_R(Q_0)) &\leq -\int_\Omega A\nabla\varphi(Y) \cdot \nabla_Y G(X,Y) - \int_\Omega B\nabla\varphi(Y)\, G(X,Y)\, \mathrm{d}Y \\
&\leq C R^{n-1} \fint_{T_{2R}(Q_0)} |\nabla_Y G(X,Y)| + C R^{n-1} \fint_{T_{2R}(Q_0)} |B(Y)| G(X,Y)\, \mathrm{d}Y \\
&\leq C R^{n-1} \left(\fint_{T_{2R}(Q)} |\nabla G(X,Y)|^2 \right)^{\frac{1}{2}} \\
&\leq C R^{n-2} G(X, A_R(Q_0)),
\end{aligned}$$

where we have used Cacciopoli's inequality and Hölder continuity up to the boundary for solutions to L^* in the last inequality. Thus the proof for the second inequality is complete. For the other inequality, observe that (3.8) gives the bound $G(X,Y) \leq C|X-Y|^{2-n}$ for all $X, Y \in \Omega$. Hence $G(X, A_R(Q_0)) \leq C R^{2-n}$ for $X \in \partial(B_{c_0 R}(A_R(Q_0)))$ for some small c_0. Theorem 2.2.8 implies that $w^X(\Delta_R(Q_0)) \geq C$ for $X \in \partial(B_{c_0 R}(A_R(Q_0)))$. Therefore

$$R^{n-2} G(X, A_R(Q_0)) \leq C w^X(\Delta_R(Q_0))$$

for all $X \in \partial(B_{c_0 R}(A_R(Q_0)))$. The maximum principle implies that the former inequality holds on $X \in \Omega \backslash B_{c_0 R}(A_R(Q_0))$. $\square$

Corollary 4.2.4 (Doubling)**.** *Let w^x be the elliptic measure for $L \in \mathcal{O}_0$. Then there exists $C > 0$ such that for $X \in \Omega \backslash T_{4R}(Q_0)$ we have*

$$w^X(\Delta_{2R}(Q_0)) \leq C w^X(\Delta_R(Q_0)).$$

Proof. Combine the Harnack inequality for the adjoint L^* and Lemma 4.2.3. $\square$

Having proven the doubling property of the elliptic measure for $L \in \mathcal{O}_0$, the Lebesgue differentiation Theorem (e.g. see [Ste93], page 13) implies that the Radon-Nikodym derivative, as defined in (4.1), is given by

$$K(X,Q) = \lim_{\Delta \searrow Q} \frac{w^X(\Delta)}{w(\Delta)}, \tag{4.3}$$

where the limit is to be understood as any sequence of surface balls Δ_j such that $Q \in \Delta_{j+1} \subset \Delta_j$ and $\bigcap_j \Delta_j = \{Q\}$. We continue with following [CFMS81] to prove a comparison Theorem for weak solutions:

Theorem 4.2.5 (Comparison Theorem)**.** *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume u and v are two non-negative weak solutions which vanish on $\Delta_{4R}(Q_0)$. Then*

$$\sup_{X \in T_R(Q_0)} \frac{u(X)}{v(X)} \leq C \frac{u(A_R(Q_0))}{v(A_R(Q_0))}.$$

Proof, compare [CFMS81]. Without losing generality, we can assume that $R \leq R_0$. If $\Omega' \subset \Omega$, then $\delta_{\Omega'}(\cdot) \leq \delta_\Omega(\cdot)$ in Ω' and so the vector field B satisfies

$$|B(X)| \leq \frac{\varepsilon_1}{\delta_\Omega(X)} \leq \frac{\varepsilon_1}{\delta_{T_{4R}(Q_0)}(X)}$$

for $X \in T_{4R}(Q_0)$. This means that the restriction of L onto $T_{4R}(Q_0)$ is in $\mathcal{O}_0$. Hence by Theorem 2.2.12 and Corollary 4.2.4, the elliptic measure $w^X_{T_{4R}(Q_0)}$ for the restriction of L onto $T_{4R}(Q_0)$ exists and is doubling.
Let $\beta = \partial T_{4R}(Q_0) \cap \Omega$ and $\alpha = \partial T_{4R}(Q_0) \cap \Omega_R$. By the doubling property we have $w^X_{T_{4R}}(\beta) \leq Cw^X_{T_{4R}}(\alpha)$ for $X \in T_R(Q_0)$. Hölder continuity up to the boundary implies that $u(X) \leq Cu(A_R(Q_0))$ for all $X \in T_R(Q_0)$, thus

$$u(X) \leq Cu(A_R(Q_0))w^X_{T_{4R}}(\beta)$$

for all $X \in \partial T_R(Q_0)$ and, by the maximum principle, for all $X \in T_R(Q)$. Harnack's inequality implies $v(A_R(Q_0)) \leq Cv(X)$ for all $X \in \alpha$, so as before, the maximum principle gives us

$$v(A_R(Q_0))w^X_{T_{4R}}(\alpha) \leq Cv(X)$$

for all $X \in T_R(Q_0)$. Thus

$$\begin{aligned} u(X) &\leq Cu(A_R(Q_0))w^X_{T_{4R}}(\beta) \\ &\leq Cu(A_R(Q_0))w^X_{T_{4R}}(\alpha) \leq Cu(V_R(Q_0))\frac{v(X)}{v(V_R(Q_0))}, \end{aligned}$$

i.e. the Comparison Theorem is proven. □

Chapter 5

The Dirichlet Problem for boundary data in $L^p(\partial\Omega)$

In Chapter 2, Theorem 2.2.12, we have seen that if $L \in \mathcal{O}$ and Ω is a Lipschitz domain, then for every $g \in C^0(\partial\Omega)$ a unique function $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ exists, such that

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \\ u &\equiv g \text{ on } \partial\Omega. \end{aligned}$$

The proof relies on the fact that one can find smooth g_j (precisely $g_j \in W^{\frac{1}{2},2}(\partial\Omega)$) which converge uniformly on $\partial\Omega$ to g. This uniform convergence in combination with the maximum principle is able to prove the solvability of the continuous Dirichlet problem.
In general, one cannot find smooth g_j which converge uniformly to g for $g \in L^p(\partial\Omega)$. Thus the method used to solve the continuous Dirichlet problem breaks down to conclude the existence of a unique weak solution $u \in W^{1,2}_{loc}(\Omega)$, which has boundary data $g \in L^p(\partial\Omega)$ in some sense.
The following chapter will deal with boundary data in $L^p(\partial\Omega)$. After a short motivation for the non-tangential maximal function, we will define the solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$ (abbreviated by $(D)_p$) and then will go on to prove some consequences of the $(D)_p$ condition. The definitions and results are well known for elliptic operators without drift terms.

5.1 Definition of the $(D)_p$ condition

Let us start this section by considering the Laplacian $L = \Delta$ on the unit ball $B = B_1(0)$. The function $P : B \times B \to [0, \infty]$ defined by

$$P(X, Y) = w_n \frac{1 - |X|^2}{|X - Y|^n},$$

where w_n is the positive constant such that $\int_{\partial B} P(X, Y) \mathrm{d}\sigma(Y) = 1$ for all $X \in B$, is the Poisson kernel for $L = \Delta$ on B (see [Fol95], page 95 for more details). For $g \in L^1(\partial B)$, let u be the Poisson integral of g, i.e.

$$u(X) = P(g)(X) = \int_{\partial B} P(X, Y) g(Y) \, \mathrm{d}\sigma(Y).$$

It is fairly obvious to see that u is harmonic in B, but less obvious if $u \equiv g$ on ∂B. In [Fol95], (2.48) Theorem, it is shown that if $g \in C^0(\partial B)$, then u extends continuously onto $\bar{B}$ and $u \equiv g$ on ∂B. Moreover, for $g \in L^p(\partial B), 1 \leq p < \infty$, it is shown that $u_r(Q) = u(rQ), 0 < r < 1,$

converges in $L^p(\partial B)$ to g for $r \to 1$. The proof makes use of the fact that the operator

$$P_r : L^p(\partial B) \to L^p(\partial B)$$
$$P_r(g) = (P(g))_r = u_r$$

is uniformly bounded in $L^p(\partial B)$ with respect to r. A similar uniform bound will allow us to conclude the almost everywhere convergence of u_r to g. We have a look at the set, where u_r does not converge to g. Choose continuous $g_j \in C^0(\partial B)$ which converge almost everywhere and in $L^p(\partial B)$ to $g \in L^p(\partial B), 1 \leq p < \infty$. Let u^j be the Poisson integral of g_j. Then we have

$$\begin{aligned}
&|\{Q \in \partial B : \lim_{r\to 1} \sup |u_r(Q) - g(Q)| > 3\varepsilon\}| \\
&\quad \leq |\{Q \in \partial B : \lim_{r\to 1} \sup |u_r(Q) - u_r^j(Q)| > \varepsilon\}| \\
&\quad + |\{Q \in \partial B : \lim_{r\to 1} \sup |u_r^j(Q) - g_j(Q)| > \varepsilon\}| \\
&\quad + |\{Q \in \partial B : \lim_{r\to 1} \sup |g_j(Q) - g(Q)| > \varepsilon\}|.
\end{aligned}$$

The third term tends to zero for $j \to \infty$ since $g_j \to g$ almost everywhere. The second term tends to zero for $j \to \infty$ since g_j is continuous and u^j is the corresponding harmonic function with boundary values g_j. In order to deal with the first term, let us assume that the operator

$$P_r^* : L^p(\partial B) \to L^p(\partial B)$$
$$P_r^*(g)(Q) = \sup_{0<r<1} (P(g))_r(Q) = \sup_{0<r<1} u_r(Q)$$

is bounded with norm M. Then Chebyshev's inequality implies

$$\begin{aligned}
|\{Q \in \partial B : \lim_{r\to 1} \sup |u_r(Q) - u_r^j(Q)| > 3\varepsilon\}| &\leq \frac{1}{(3\varepsilon)^p} \int_{\partial B} |u_r(Q) - u_r^j(Q)|^p \, \mathrm{d}Q \\
&\leq \frac{M^p}{(3\varepsilon)^p} ||g - g_j||^p_{L^p(\partial B)}.
\end{aligned}$$

Thus, under the assumption that the operator P_r^* is bounded we can conclude that u_r converges almost everywhere and in $L^p(\partial B)$ to g for $g \in L^p(\partial B)$. Therefore we will define the solvability of the Dirichlet problem with boundary data in L^p (the $(D)_p$ condition) by a similar assumption as the boundedness of P_r^*. Let us mention two more motivating ideas for the $(D)_p$ condition:

- as it was pointed out in Chapter 4, for general $L \in \mathcal{O}$ and Ω a Lipschitz domain, the Poisson kernel might not exist. Thus, in general, we cannot speak about the operator P_r. But since the continuous Dirichlet problem is solvable for $L \in \mathcal{O}$, one can speak about the "equivalent" of the operator P_r if restricted to continuous boundary data, i.e. the map that maps the function of continuous boundary data to its weak solution.
- if a weak solution is non-negative, then by the Harnack's principle the weak solution is essentially constant on $B_{\frac{\delta(X)}{2}}(X)$ balls. Thus, instead of taking the supremum over the straight line $\{rQ : 0 < r < 1\}$ in the definition of P_r^*, one can take the supremum over a non-tangential approach region, which we will define now:

Definition 5.1.1. *Let Ω be a Lipschitz domain. For $\kappa > 1$ we define the cone-like family of non-tangential approach regions $\{\Gamma_\kappa(Q)\}_{Q\in\partial\Omega}$ by*

$$\Gamma_\kappa(Q) = \{X \in \Omega : |X - Q| < \kappa \, dist(X, \partial\Omega)\}.$$

The non-tangential maximal function for the non-tangential approach region $\{\Gamma_\kappa(Q)\}_{Q\in\partial\Omega}$ is defined by

$$u_\kappa^*(Q) = \sup_{X\in\Gamma_\kappa(Q)} |u(X)|.$$

If no confusion is possible, we will omit the index κ and we will write u^* instead of u_κ^* and $\Gamma(Q)$ instead of $\Gamma_\kappa(Q)$. The truncation at height h of the non-tangential maximal function is *denoted by* $(u)^*_h(Q) = \sup_{X\in\Gamma(Q)\cap B_h(Q)} |u(X)|$.

Having defined the non-tangential maximal function, we now define what is meant by the solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$, which is the well known definition of the solvability for elliptic operators without drift terms (see e.g. [Ken94]).

Definition 5.1.2. *Let* Ω *be a Lipschitz domain and* $L \in \mathcal{O}$. *We say that the Dirichlet problem for* L *with boundary data in* $L^p(\partial\Omega), 1 < p < \infty$, *is solvable (abbreviated* $(D)^L_p$*), if there exists a constant* $C > 0$ *such that for every* $g \in C^0(\partial\Omega)$, *the corresponding unique weak solution* $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ *satisfies*[1]

$$||u^*||_{L^p(\partial\Omega)} \leq C||g||_{L^p(\partial\Omega)},$$

where the constant C *depends on the aperture of the non-tangential maximal function. This dependence will be irrelevant and is therefore omitted in the notation. If no confusion can arise, we omit the index* L *and we write* $(D)_p$. *Moreover, we define* $(D^*)_p = (D)^{L^*}_p$, *where* L^* *is the adjoint of* L.

In the next theorem, we show that the $(D)_p$ condition allows us to conclude that for every $f \in L^p(\partial\Omega)$ there exists a function $u \in W^{1,2}_{loc}(\Omega)$ such that u converges non-tangentially to f almost everywhere. This result is well known for elliptic operators without drift terms and a proof can be found in [Ken94]. We will include a detailed proof for completeness.

Theorem 5.1.1. *Let* $L \in \mathcal{O}$ *and* Ω *be a Lipschitz domain. Assume that* $(D)_p$ *holds, then for every* $g \in L^p(\partial\Omega)$, *there exists* $u \in W^{1,2}_{loc}(\Omega)$ *with* $||u^*||_{L^p(\partial\Omega)} \leq C_{(D)_p}||g||_{L^p(\partial\Omega)}$ *such that* $Lu = 0$ *in* Ω *and that* u *converges non-tangentially to* g *almost everywhere.*

Proof. Choose $g_j \in C^0(\partial\Omega)$ with $g_j \to g$ in $L^p(\partial\Omega)$ and almost everywhere. Let u_j be the unique weak solution with boundary data g_j. We claim that for a fixed compact $K \subset \Omega$, u_j converges uniformly on K to some function u.
For this, we first show that for a fixed $X \in \Omega$, the sequence $u_j(X)$ is a Cauchy sequence. Choose $\varepsilon_2 > 0$ and then $N = N(\varepsilon_2)$ such that $||g_j - g_k||_{L^p(\partial\Omega)} < \varepsilon_2$ for all $j, k \geq N$. Since Ω is a Lipschitz domain there exist $Q_0 \in \partial\Omega$ and $\alpha \approx \delta(X)$, such that $X \in \Gamma(Q)$ for all $Q \in \Delta_\alpha(Q_0)$. Hence $|u_j(X) - u_k(X)| \leq (u_j - u_k)^*(Q)$ for all $Q \in \Delta_\alpha(Q_0)$ and therefore

$$\begin{aligned} |u_j(X) - u_k(X)| &\leq \fint_{\Delta_\alpha(Q_0)} (u_j - u_k)^*(Q)\, \mathrm{d}\sigma(Q) \\ &\leq C_{(\alpha,(D)_p)}||g_j - g_k||_{L^p(\partial\Omega)} \leq C_{(\alpha,(D)_p)}\varepsilon_2. \end{aligned} \tag{5.1}$$

Thus the sequence $\{u_j(X)\}_j$ is a Cauchy sequence and $u_j(X)$ converges to some $u(X)$. By considering $(g_j - g_k)_+$ and $(g_j - g_k)_-$ and repeating the previous argument, the Harnack inequality implies that the convergence of u_j to u is uniform on K. Thus, Lemma 4.2.1 implies $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and that u is a weak solution.
Let $(u)^{\varepsilon,*} = \sup_{X\in\Gamma(Q)\setminus B_\varepsilon(Q)} |u(X)|$ for some small but fixed $\varepsilon > 0$. Then

$$||(u)^{\varepsilon,*}||_{L^p(\partial\Omega)} \leq ||(u - u_j)^{\varepsilon,*}||_{L^p(\partial\Omega)} + ||(u_j)^{\varepsilon,*}||_{L^p(\partial\Omega)}.$$

The first term tends towards zero for fixed $\varepsilon > 0$ and $j \to \infty$ since $u_j \to u$ uniformly on any compact $K \subset \Omega$. For the second term we have $||(u_j)^{\varepsilon,*}||_{L^p(\partial\Omega)} \leq ||u^*_j||_{L^p(\partial\Omega)} \leq C||g_j||_{L^p(\partial\Omega)}$. Therefore $||(u)^{\varepsilon,*}||_{L^p(\partial\Omega)} \leq C||g||_{L^p(\partial\Omega)}$ uniformly in ε. The monotone convergence theorem implies

$$||u^*||_{L^p(\partial\Omega)} = \lim_{\varepsilon\to 0} ||(u)^{\varepsilon,*}||_{L^p(\partial\Omega)} \leq C_{(D)_p}||g||_{L^p(\partial\Omega)}$$

Next we claim that $||(u - u_j)^*||_{L^p(\partial\Omega)} \to 0$ for $j \to \infty$. By the monotone convergence theorem we have

$$||(u - u_j)^*||_{L^p(\partial\Omega)} = \lim_{\varepsilon\to 0} ||(u - u_j)^{\varepsilon,*}||_{L^p(\partial\Omega)}.$$

[1] It is well known that if $||u^*_\kappa||_{L^p(\partial\Omega)} < \infty$ for some aperture κ, then $||u^*_{\kappa'}||_{L^p(\partial\Omega)} \approx ||u^*_\kappa||_{L^p(\partial\Omega)}$ for any aperture κ', where the implicit constant depends on κ and κ'.

Since $u_k \to u$ uniformly on compact subsets and $||(u_k - u_j)^{\varepsilon,*}||_{L^p(\partial\Omega)} \le ||(u_k - u_j)^*||_{L^p(\partial\Omega)}$ we get

$$\begin{aligned}||(u-u_j)^*||_{L^p(\partial\Omega)} &\le \lim_{\varepsilon\to 0}\lim_{k\to\infty} ||(u-u_k)^{\varepsilon,*}||_{L^p(\partial\Omega)} + \lim_{k\to\infty} ||(u_k-u_j)^*||_{L^p(\partial\Omega)} \\ &\le C||g-g_j||_{L^p(\partial\Omega)},\end{aligned}$$

hence $\lim_{j\to\infty} ||(u-u_j)^*||_{L^p(\partial\Omega)} \to 0$. It remains to be shown that u converges non-tangentially to g almost everywhere. For this we show that the set, where u does not converge non-tangentially to g, has measure zero. We have

$$\begin{aligned}|\{Q\in\partial\Omega : \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} |u(X)-g(Q)| > 3\varepsilon\}| \le{}& |\{Q\in\partial\Omega : \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} |u(X)-u_j(Q)| > \varepsilon\}| \\ &+ |\{Q\in\partial\Omega : \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} |u_j(X)-g_j(Q)| > \varepsilon\}| \\ &+ |\{Q\in\partial\Omega : \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} |g_j(Q)-g(Q)| > \varepsilon\}|\end{aligned}$$

and therefore by Chebyshev's inequality

$$|\{Q\in\partial\Omega : \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} |u(X)-g(Q)| > 3\varepsilon\}| \le \lim_{j\to\infty} \frac{1}{\varepsilon^p} ||(u-u_j)^*||^p_{L^p(\partial\Omega)},$$

which implies that u converges non-tangentially to g almost everywhere. □

Remark 5.1.2. *We would like to point out that in Theorem 5.1.1 nothing is said about the uniqueness of* u*. For the proof of the uniqueness result, we will need the existence of the Green's function and the reverse Hölder property of the elliptic measure, for which we have to restrict ourselves to* $L \in \mathcal{O}_0$*. The proof will be carried out in Theorem 5.3.5.*

Let us summarise a few important results on the study of the Dirichlet problem with boundary data in L^p and therefore on the study of the $(D)_p$ condition: the study began in [Dah77] and [Dah79]. B.E.J. Dahlberg proves that for $L = \Delta$ and Ω a Lipschitz domain, the $(D)_p^\Delta$ condition holds for $2 \le p < \infty$ and that if Ω is a C^1 domain, then one gets the range $1 < p < \infty$. In [FJK84], elliptic operators of the form $L = \mathrm{div}A\nabla$ for symmetric A on a bounded C^1 domain are considered. The authors show that $(D)_p^L$ holds for $2 \le p < \infty$ if the modulus of continuity of A in the direction of a transverse vector field satisfies an integrability condition. The symmetry assumption on A plays a major part in the proof. In [KKPT00], the study of elliptic operators $L_0 = \mathrm{div}A\nabla$ for possibly non-symmetric coefficients is initiated. It is shown that, if one lives in two dimensions and if A depends only on one of the variables, the $(D)_p$ condition holds for $p \in (q, \infty)$ and some $q < \infty$. In the papers [KP01] and [DPP07] elliptic operators with drift terms $L = \mathrm{div}A\nabla u + B\nabla u$ are studied. Under a Carleson measure type assumption on A and B, the solvability of $(D)_p$ for some (possibly large) p is achieved.
All the results mentioned above prove the solvability under the assumption of some restrictions on the coefficients. In contrast to those results stand the perturbation results: here, it is assumed that $(D)_p^{L_0}$ holds for an elliptic operator L_0 and then proven that $(D)_q^{L_1}$ holds for another elliptic operator L_1, which might be seen as a perturbation of L_0. This was done for operators without drift terms for example in [Dah86], [Fef89] and [FKP91], where different Carleson measure type perturbations are considered.
Amongst the results, which prove solvability of $(D)_p$ directly or by perturbation, are results which give consequences of the $(D)_p$ condition, e.g. see [DKP10], where an extrapolation property for the $(D)_p$ condition at the endpoint ∞ (see Theorem 5.3.6) is proven. This endpoint extrapolation property is the main interest of Chapter 6; not for the Dirichlet problem, but for the regularity problem.

5.2 A_p weights and the reverse Hölder class B_p

For proving or dealing with the $(D)_p$ condition, the class of A_p weights of Muckenhoupt[2] will be useful. We will see how the elliptic measure for an elliptic operator $L \in \mathcal{O}_0$, the $(D)_p$ condition and the A_p class are related.
In the first part of this section, we lay down the basics of the A_p weights and of the reverse Hölder class B_q (detailed introductions can be found in [DCU01], [Gra09] and [Jou83]). In the second part, we will give a detailed proof of a $\mathrm{L}\log\mathrm{L}$–characterization of A_∞, which is the endpoint of Gehring's Lemma and is well-known, but not contained in the above mentioned monographs.
Only in this section we will use a slightly different notation, than in the rest of the thesis. We apologise for it, but we thought that it is worth keeping the notation used in the literature for the study of the A_p weights. Thus, in this section the letter Q is reserved for cubes in $\mathbb{R}^n$.

5.2.1 Some Basics of the classes A_p and B_p

A function k is called a weight if it is non-negative and locally integrable. We define the Muckenhoupt class of A_p weights for $1 < p < \infty$ by the class of weights k for which

$$[k]_{A_p} = \sup_Q \left(\fint_Q k \,\mathrm{d}x \right) \left(\fint_Q k^{1-p'} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q and p' denotes the conjugate index of p as usual. The constant $[k]_{A_p}$ is called the A_p-constant of k. In [Muc72], it is proven that the class of A_p-weights is the class of weights for which the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \,\mathrm{d}y,$$

is bounded on the weighted spaces $L^p(\mathbb{R}^n, k\,\mathrm{d}x)$, i.e. $k \in A_p$ for $1 < p < \infty$ if and only if

$$||Mf||_{L^p(k\,\mathrm{d}x)} \leq C||f||_{L^p(k\,\mathrm{d}x)}.$$

The class A_∞ is defined by all weights k such that

$$[k]_{A_\infty} = \sup_Q \left[\left(\fint_Q k \right) \exp\left(\fint_Q \log k^{-1} \right) \right] < \infty.$$

Furthermore, we define the reverse Hölder class of index $p, 1 < p < \infty$, denoted by B_p, by all weights k such that the reverse Hölder condition

$$\left(\fint_Q k^p \right)^{\frac{1}{p}} \leq C \fint_Q k$$

holds for all cubes Q and a constant $C \geq 1$ independent of Q. For a weight k and $f \in L^1_{loc}(\mathbb{R}^n)$, let

$$M_k f(x) = \sup_{Q \ni x} \frac{1}{k(Q)} \int_Q |f(x)| k(x) \,\mathrm{d}x,$$

where $k(Q) = \int_Q k \,\mathrm{d}x$. Then we have

Theorem 5.2.1. *Let $1 < p < \infty$:*

- *$k \in A_p$ implies $k \in B_q$ for some $1 < q < \infty$.*
- *$k \in B_p$ implies $k \in A_q$ for some $1 < q < \infty$.*
- *for $k \in A_p$ there exists $\varepsilon > 0$ such that $k \in A_{p-\varepsilon}$.*

[2] These classes originate in [Ros62].

- *Gehring's Lemma, [Geh73]: for $k \in B_p$ there exists $\varepsilon > 0$ such that $k \in B_{p+\varepsilon}$.*
- *for $k \in A_\infty$ there exists $q < \infty$ such that $k \in A_q$.*
- *$k \in B_p$ if and only if M_k is a bounded operator from $L^{p'}(\,dx)$ to $L^{p'}(\,dx)$.*

Proof. We will prove only the last assertion. The proofs of the other assertions can be found in [Gra09]. We claim that $k \in B_p$ if and only if $k^{-1} \in A_{p'}(k\,\mathrm{d}x)$, where $A_{p'}(k\,\mathrm{d}x)$ is defined as A_p but with the Lebesgue measure replaced by the measure $k\,\mathrm{d}x$. The statement $k^{-1} \in A_p(k\,\mathrm{d}x)$ means that

$$\left(\frac{1}{k(Q)}\int_Q k^{-1}k\right)\left(\frac{1}{k(Q)}\int_Q k^{p-1}k\right)^{p'-1} \le C$$

for some $C > 0$ and all cubes Q. Since $p' - 1 = \frac{p'}{p}$, we get

$$\frac{|Q|}{k(Q)}\left(\frac{1}{k(Q)}\int k^p\right)^{\frac{p'}{p}} \le C.$$

Thus

$$\frac{|Q|}{k(Q)}\left(\frac{1}{|Q|}\int k^p\right)^{\frac{1}{p}} \le C^{\frac{1}{p'}}$$

for all cubes Q, which is the reverse Hölder condition. Since all steps we took are invertible, the claim is proven. Next, we claim that $k^{-1} \in A_{p'}(k\,\mathrm{d}x)$ is equivalent to

$$||M_k f||_{L^{p'}(\,\mathrm{d}x)} \le C||f||_{L^{p'}(\,\mathrm{d}x)}. \tag{5.2}$$

This follows from the fact that $k^{-1} \in A_{p'}(k\mathrm{d}x)$ is equivalent to the boundedness of the maximal function defined with respect to the measure $k\,\mathrm{d}x$, which is M_k, on the weighted $L^{p'}$-spaces with measure $k^{-1}k\,\mathrm{d}x = \,\mathrm{d}x$, i.e. precisely (5.2). ☐

5.2.2 A $\mathrm{L}\log\mathrm{L}$–characterization of A_∞

In the books [DCU01], [Gra09] and [Jou83] one can find several characterizations of the class A_∞ with the corresponding proofs. The following Theorem is a summary of these characterizations:

Theorem 5.2.2. *For a weight k, the following are equivalent:*

- *$k \in A_\infty$*
- *$k \in B_q$ for some $q > 1$*
- *there exist $0 < \alpha, \beta < 1$ such that for all cubes Q and all measurable subsets E of Q, $|E| \le \alpha|Q|$ implies $k(E) \le \beta k(Q)$.*
- *there exist $0 < \alpha, \beta < 1$ such that for all cubes Q and all measurable subsets E of Q, $k(E) \le \alpha k(Q)$ implies $|E| \le \beta|Q|$.*
- *there exist $0 < C, \varepsilon < \infty$ such that for all cubes Q and all measurable subsets E of Q, we have*
$$\frac{k(E)}{k(Q)} \le C\left(\frac{|E|}{|Q|}\right)^{\varepsilon}.$$

To this list, we will add a $\mathrm{L}\log\mathrm{L}$–characterization, which is the endpoint of Gehring's Lemma. This $\mathrm{L}\log\mathrm{L}$–characterization is well known and for example used by R. Fefferman in [Fef89], equation (5), in order to prove a perturbation result, as mentioned in section 5.1.
I am grateful to J. Pipher, who explained the main structure of the following proof to me based on Young's inequality of Orlicz spaces.
Proofs based on a real method of interpolation can be found in [Mil96] and [BMR99], and a more general proof than ours can be found in [Buc93].
We start by defining Young functions and the corresponding Orlicz spaces. A function Φ, which

is continuous, increasing, convex, 0 at 0 and $\lim_{t\to\infty}\Phi(t)=\infty$, is called a Young function. On a measure space (X,μ), the Orlicz norm $||\cdot||_{\Phi(L)(X,\mu)}$ for the Young function Φ is defined by

$$||f||_{\Phi(L)(X,\mu)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|f|}{\lambda}\right) \mathrm{d}\mu \leq 1\}.$$

The Orlicz space for the Young function Φ is the collection of all measurable functions with finite Orlicz norm $||\cdot||_{\Phi(L)(X,\mu)}$ (for more details see [Gra09] page 158 - page 165). For example, the Young function $\Phi(t)=t^p, 1\leq p<\infty$ gives the usual $L^p(X,\mathrm{d}\mu)$ spaces. The following Lemmata are left as an exercise in [Gra09]. For completeness, we include the proofs, since they will lead to the L log L–characterization of A_∞.

Lemma 5.2.3 (Young's inequality for Orlicz spaces). *Let φ be a continuous strictly increasing function on $[0,\infty)$ with $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Define $\Phi(x)=\int_0^x\varphi(t)\,dt$ and $\Psi(x)=\int_0^x\varphi^{-1}(t)\,dt$. Then, for $s,t\in[0,\infty)$, one has*

$$st \leq \Phi(s)+\Psi(t). \tag{5.3}$$

Proof. For the proof fix s and let $t_0=\varphi(s)$. We distinguish two cases, namely $t_0\leq t$ and $t_0\geq t$. We deal only with the case $t_0\leq t$. The case $t_0\geq t$ can be proven analogously.
Interpreting the product $u\varphi(u), u\geq 0$ as the area of the rectangle $u\times\varphi(u)$, we see that

$$\int_0^u \varphi(t)\,\mathrm{d}t + \int_0^{\varphi(u)} \varphi^{-1}(t)\,\mathrm{d}t = u\varphi(u),$$

since the first term on the left hand side is the lower right part of the mentioned rectangle and the second term the top-left part. This shows that

$$s\varphi(s) = st_0 = \int_0^s \varphi(r)\,\mathrm{d}r + \int_0^{\varphi(s)} \varphi^{-1}(t)\,\mathrm{d}t = \Phi(s)+\Psi(t_0),$$

i.e. that equality holds in (5.3). For any $r\in[t_0,t]$, we have $\varphi^{-1}(r)\geq s$ and so

$$st - st_0 = \int_{t_0}^t s\,\mathrm{d}r \leq \int_{t_0}^t \varphi^{-1}(r)\,\mathrm{d}r = [\Phi(s)+\Psi(t)] - [\Phi(s)+\Psi(t_0)].$$

Since $st_0=\Phi(s)+\Psi(t_0)$, we get (5.3). □

Corollary 5.2.4. *For $s,t\in[0,\infty)$, we define the Young functions Φ, respectively Ψ, by $\Phi(t)=t\log(t+1)$, respectively $\Psi(s)=e^s-1$, then*

$$st \leq \Phi(t)+\Psi(s).$$

The Orlicz norms corresponding to Φ, respectively Ψ, will be denoted by $||\cdot||_{L\log L}$, respectively $||\cdot||_{\exp L}$.

Proof. It is easy to check that Φ and Ψ are Young functions. Let $\varphi(t)=\log(t+1)$. Then φ satisfies the assumptions of the Lemma for Young's inequality for Orlicz spaces. For this φ, we have $\int_0^x\varphi(t)\,\mathrm{d}t=(t+1)\log(t+1)-(t+1)$ and $\int_0^x\varphi^{-1}(s)\,\mathrm{d}s=\int_0^x e^s-1\,\mathrm{d}s=e^s-s-1$. Thus Young's inequality for Orlicz spaces implies

$$st \leq (t+1)\log(t+1)-t+e^s-s-1 \leq \Phi(t)+\Psi(s).$$

□

Lemma 5.2.5 (Hölder's inequality for Orlicz spaces). *Let Φ and Ψ be two Young functions such that (5.3) holds, then*

$$|\langle f,g\rangle| \leq 2||f||_{\Phi(L)}||g||_{\Psi(L)}.$$

Proof. Without losing generality, we can assume that $f \in \Phi(L)(X,\mu)$ and $g \in \Psi(L)(X,\mu)$. By (5.3), we know that for every $0 < \lambda, \nu < \infty$

$$|\int fg| \leq \lambda\nu \int \left[\Psi\left(\frac{|f|}{\lambda}\right) + \Psi\left(\frac{|g|}{\nu}\right)\right]$$

holds. Choosing $\lambda = ||f||_{\Phi(L)}$ and $\nu = ||g||_{\Psi(L)}$, we see – by the definition of the Orlicz norm – that the integral on the right hand side is at most 2 and so, the proof is complete. □

Theorem 5.2.6. *Let $M_Q f(x) = \sup_{\substack{R \ni x \\ R \subset Q}} \fint_R |f(z)|\, dz$, where R and Q are cubes, then*

$$||f||_{L\log L(Q, \frac{dx}{|Q|})} \leq c_n ||M_Q f||_{L^1(Q, \frac{dx}{|Q|})}.$$

We will use the proof found in [Gra09], Lemma 7.5.4. to prove it. The crucial step for the proof of Theorem 5.2.6 is the fact that one can reverse the weak-type (1,1) inequality (see [Ste70], page 5)

$$|\{M_Q f > \lambda\}| \leq \frac{c_n}{\lambda} \int_{\{f > \frac{\lambda}{2}\} \cap Q} |f|\, \mathrm{d}x, \tag{5.4}$$

which can be derived from the well known weak $(1,1)$ inequality as follows:
Let $0 \leq f \in L^1(Q)$ and $f_1 = f\chi_{\{f > \frac{\lambda}{2}\}}$, then $f \leq f_1 + \frac{\lambda}{2}$ and $M_Q f \leq M_Q f_1 + \frac{\lambda}{2}$. Thus

$$|\{M_Q f > \lambda\}| \leq |\{M_Q f_1 > \frac{\lambda}{2}\}| \leq \frac{C}{\lambda} \int_Q f_1 = \frac{C}{\lambda} \int_{\{f > \frac{\lambda}{2}\} \cap Q} f.$$

Lemma 5.2.7. *For $f \in L^1(Q, \frac{dx}{|Q|})$ and any $\lambda > \fint_Q |f|$, there exists a c_n, such that*

$$\frac{1}{\lambda} \int_{\{|f| \geq \lambda\} \cap Q} |f|\, dx \leq c_n |\{x \in Q : M_Q f > \lambda\}|. \tag{5.5}$$

Proof. The proof is taken form [Ste69]. We apply a Calderón–Zygmund decomposition to $|f|$ at the height λ. This gives a sequence of essentially disjoint[3] cubes $\{Q_j\}_j \subset Q$ such that

$$\lambda < \fint_{Q_j} |f(x)|\, \mathrm{d}x \leq 2^n \lambda$$

and $|f(x)| < \lambda$ for $x \in Q \backslash \bigcup_j Q_j$. Therefore, $M_Q f(x) > \lambda$ for $x \in Q_j$ and

$$|\{M_Q f > \lambda\}| \geq c_n \sum_j |Q_j| \geq \frac{c_n}{\lambda} \int_{\bigcup_j Q_j} |f|\, \mathrm{d}x \geq \frac{c_n}{\lambda} \int_{\{|f| \geq \lambda\} \cap Q} |f|\, \mathrm{d}x,$$

which completes the proof. □

Proof of Theorem 5.2.6. By the definition of the Orlicz norm, it is enough to show that for $\lambda_Q = c||M_Q f||_{L^1(Q, \frac{dx}{|Q|})}$ with c large enough (depending only on the dimension), one gets

$$\fint_Q \frac{|f|}{\lambda_0} \log\left(1 + \frac{|f|}{\lambda_0}\right) \mathrm{d}x \leq 1.$$

[3] By essentially disjoint we mean that $\sum_j \chi_{Q_j}(x) \leq C$ for all $x \in Q$ and a constant C depending only on the dimension, where χ_{Q_j} denotes the characteristic function of Q_j.

Let $h = \frac{|f|}{\lambda_0}$ and $h_Q = \fint_Q h$, then

$$\fint_Q h(x)\log(1+h(x))\,\mathrm{d}x = \frac{1}{|Q|}\int_0^{h_Q}\frac{1}{1+t}\int_{Q\cap\{h>t\}} h(x)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|Q|}\int_{h_Q}^{\infty}\frac{1}{1+t}\int_{Q\cap\{h>t\}} h(x)\,\mathrm{d}x\,\mathrm{d}t = I + II.$$

For the first integral, we get

$$I \le \frac{1}{|Q|}\int_0^{h_Q}\int_Q h(x)\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\lambda_0^2}\left(\fint_Q f\right)^2 \le \frac{1}{\lambda_0^2}||M_Q f||^2_{L^1(Q,\frac{\mathrm{d}x}{|Q|})} = \frac{1}{c^2}.$$

For the second term, we apply Lemma 5.2.7 to get

$$II \le \frac{c_n}{|Q|}\int_{h_Q}^{\infty}\frac{t}{t+1}|\{M_Q h > t\}|\,\mathrm{d}t \le c_n \fint_Q M_Q h\,\mathrm{d}x = \frac{c_n}{\lambda_Q}\fint_Q M_Q f\,\mathrm{d}x = \frac{c_n}{c}.$$

Thus, if we choose c large enough such that $\frac{c_n}{c} + \frac{1}{c^2} \le 1$ holds, we get $\fint_Q \frac{|f|}{\lambda_0}\log(1+\frac{|f|}{\lambda_0})\,\mathrm{d}x \le 1$, which completes the proof of the Theorem. ☐

The next Lemma we need is a result from [CR80]:

Lemma 5.2.8. *Let $h \in L^1_{loc}$ such that $Mh < \infty$ almost everywhere, then*

$$||\log M(h)||_{BMO} \le C,$$

where C depends only on the dimension.

For the proof, we need some knowledge about A_1 and BLO (bounded lower oscillation). We say that a function $b \in L^1_{loc}$ is in BLO if there exists a constant c such that

$$b_Q - \inf_Q b \le c \tag{5.6}$$

for all cubes Q. The smallest c for which (5.6) holds is the BLO norm (modulo constants) of b. For $b \in L^1_{loc}$, we have $\int_Q (b-b_Q)_+ = -\int_Q (b-b_Q)_-$ and therefore

$$\frac{1}{|Q|}\int_Q |b-b_Q| = \frac{2}{|Q|}\int_Q (b_Q - b)_+ \le b_Q - \inf_Q b.$$

Thus $||b||_{\mathrm{BMO}} \le 2||b||_{\mathrm{BLO}}$.
A weight k is in Muckenhoupt's class A_1 (see [Ste93] and the references mentioned there) if and only if there exists $c > 0$ such that

$$k_Q \le c \inf_Q k$$

for all cubes Q, i.e. if and only if $Mk \le ck$. The smallest c with which this inequality holds is the A_1-constant of k, which is denoted by $[k]_{A_1}$.

Proposition 5.2.9. *If $e^{\gamma f} \in A_1$, then $f \in BLO$ with $||f||_{BLO} \le \frac{\log[e^{\gamma f}]_{A_1}}{\gamma}$.*

Proof. The proof is taken from the proof of Lemma 1 in [CR80]. By the A_1-condition, we have $(e^{\gamma f})_Q \le [e^{\gamma f}]_{A_1}\inf_Q e^{\gamma f}$. Jensen's inequality gives us

$$(\gamma f)_Q \le \log(e^{\gamma f})_Q \le \log[e^{\gamma f}]_{A_1} + \gamma \inf_Q f.$$

Thus, $f_Q - \inf_Q f \le \frac{\log[e^{\gamma f}]_{A_1}}{\gamma}$. ☐

The following can be found in [Ste93], V 5.2:

Proposition 5.2.10. *Let $0 < \delta < 1$. Assume that μ is a locally finite positive Borel measure with $M\mu < \infty$ almost everywhere. Then $(M\mu)^\delta \in A_1$ with $[(M\mu)^\delta]_{A_1} \leq c_{n,\delta}$, i.e. the A_1-constant is uniformly bounded with respect to μ.*

Proof. Fix Q. Let $\bar{Q}$ be the concentric double of Q. We split the measure μ into $\chi_{\bar{Q}}\mu + \chi_{\bar{Q}^c}\mu = \mu_1 + \mu_2$. Since $(M\mu)^\delta \leq (M\mu_1)^\delta + (M\mu_2)^\delta$ it suffices to prove that $((M\mu_j)^\delta)_Q \leq C \inf_Q (M\mu)^\delta$ for $j = 1, 2$. For μ_1, we use Kolmogorov's inequality (see e.g. [Gra08], page 91) to get

$$\frac{1}{|\bar{Q}|} \int_{\bar{Q}} (M\mu_1)^\delta \leq c_{n,\delta} |\bar{Q}|^{-\delta} \left(\int_{R^n} \mu_1 \right)^\delta .$$

Since μ_1 is supported on $\bar{Q}$, we get $((M\mu_1)^\delta)_{\bar{Q}} \leq c_{n,\delta} ((\mu_1)_{\bar{Q}})^\delta \leq c_{n,\delta} \inf_{\bar{Q}} (M\mu)^\delta$.
For μ_2, we use the fact that for $x, y \in Q$, $M\mu_2(x) \leq c_n M\mu_2(y)$ holds. Hence

$$(M\mu_2)^\delta(x) \leq c_n \inf_Q (M\mu)^\delta .$$

An integration over Q shows that $((M\mu_2)^\delta)_Q \leq \inf_Q (M\mu)^\delta$ and so the proof is complete. □

Proof of Lemma 5.2.8. By Proposition 5.2.10, $e^{\frac{1}{2}\log Mh} = Mh^{\frac{1}{2}} \in A_1$ and the corresponding A_1-constant can be bounded by a constant depending only on the constant. Hence Proposition 5.2.9 implies

$$|| \log Mh ||_{\mathrm{BMO}} \leq 2 || \log Mh ||_{\mathrm{BLO}} \leq c_n .$$

□

Finally, we can prove the endpoint of Gehring's Lemma, i.e. the A_∞–characterization, in terms of an $\mathrm{L} \log \mathrm{L}$–estimate.

Theorem 5.2.11. *Let $k \in L^1_{loc}$ be a non-negative function. Then $k \in A_\infty$ if and only if there exists a constant $C > 0$ such that*

$$||k||_{L \log L(Q, \frac{dx}{|Q|})} \leq C ||k||_{L^1(Q, \frac{dx}{|Q|})} \tag{5.7}$$

for all cubes Q.

Proof. Assuming $k \in A_\infty$ first, we know that k satisfies a reverse Hölder inequality of order γ for some $\gamma > 1$. Theorem 5.2.6 implies

$$\begin{aligned} ||k||_{\mathrm{L}\log\mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} &\leq c_n ||M_Q k||_{L^1(Q, \frac{\mathrm{d}x}{|Q|})} \\ &\leq c_n ||M_Q k||_{L^\gamma(Q, \frac{\mathrm{d}x}{|Q|})} \\ &\leq C_\gamma ||k||_{L^\gamma(Q, \frac{\mathrm{d}x}{|Q|})} \leq C ||k||_{L^1(Q, \frac{\mathrm{d}x}{|Q|})}, \end{aligned}$$

and so (5.7) holds.
Now we assume that (5.7) holds. Then, it is enough to show by Theorem 5.2.2 that there exists $0 < \eta, \beta < 1$ such that for any measurable $E \subset Q$ with $|E| \leq \eta |Q|$, we have $k(E) \leq \beta k(Q)$ where $k(E) = \int_E k(x)\,\mathrm{d}x$. For this, let $f = \max(0, 1 + \delta \log M(\chi_E))$ as in [JJ94] and [DKP10] for some $\delta > 0$ to be determined later. Then f has the following properties:

- $0 \leq f \leq 1$
- $f = 1$ a.e. on E
- $||f||_{\mathrm{BMO}} \leq c_n \delta$, which is implied by $|| \max(f, g) ||_{\mathrm{BMO}} \leq c||f||_{\mathrm{BMO}} + c||g||_{\mathrm{BMO}}$ and Lemma 5.2.8

Thus we have

$$\begin{aligned} k(E) &\le \int_Q fk \, \mathrm{d}x \\ &= \int_Q (f - f_Q)k \, \mathrm{d}x + f_Q \int_Q k \, \mathrm{d}x = I + II. \end{aligned}$$

We deal with II first. Observe that $\{1 + \delta \log M(\chi_E) > 0\} = \{M(\chi_E) > e^{-\frac{1}{\delta}}\}$. Thus $|\{f > 0\}| \le c_n e^{\frac{1}{\delta}} |E|$, from which it follows that $⨍_Q f \le c_n e^{\frac{1}{\delta}} \frac{|E|}{|Q|} \le c_n e^{\frac{1}{\delta}} \eta$ and so

$$II \le c_n e^{\frac{1}{\delta}} \eta k(Q).$$

For I, we use Hölder's inequality for Orlicz spaces to get

$$I \le 2||f - f_Q||_{\exp \mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} ||k||_{\mathrm{L} \log \mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} \le C||f - f_Q||_{\exp \mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} k(Q).$$

We claim that $||f - f_Q||_{\exp \mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} \le c_n ||f||_{\mathrm{BMO}}$. To prove the claim, it is sufficient to show that for $\lambda_0 = c_0 2^n e ||f||_{\mathrm{BMO}}$, with $c_0 > 1$ to be determined later and depending on the dimension, we have

$$⨍_Q \exp\left(\frac{|f - f_Q|}{\lambda_0}\right) - 1 \, \mathrm{d}x \le 1.$$

Since $c_0 > 1$, we can apply the John–Nirenberg inequality (see Appendix, Theorem A.0.29), to get

$$⨍_Q \exp\left(\frac{|f - f_Q|}{\lambda_0}\right) - 1 \, \mathrm{d}x \le \frac{e c_0^{-1}}{1 - c_0^{-1}},$$

which is smaller than 1 for c_0 large enough. Thus $||f - f_Q||_{\exp \mathrm{L}(Q, \frac{\mathrm{d}x}{|Q|})} \le c_n ||f||_{\mathrm{BMO}}$ and so the claim is proven.

Gathering the estimates for I and II together and using the fact that $||f||_{\mathrm{BMO}} \le c_n \delta$, we see that

$$k(E) \le I + II \le c_n \delta k(Q) + c_n e^{\frac{1}{\delta}} \eta k(Q) = \beta k(Q).$$

First, we choose δ such that $\delta \le \frac{1}{3c_n}$ and then we choose η such that $\eta \le (3c_0 e^{\frac{1}{\delta}})^{-1}$, which gives us $\beta \le \frac{2}{3}$ and so, the theorem is proven. $\square$

5.3 Consequences of the $(D)_p$ condition

In this section, we will see that the $(D)_p$ condition for an elliptic operator in $\mathcal{O}_0$ is equivalent to a reverse Hölder condition of index p' for the elliptic measure. This will allow us to answer the uniqueness question which was left open in Theorem 5.1.1. We extend a result from [Ken94], (Corollary 1.3.8) for elliptic operators without drift terms to elliptic operators in $\mathcal{O}_0$.

Lemma 5.3.1. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. For any $s > 0, R \le R_0$, where R_0 depends on the domain Ω, see Lemma 2.1.1, $Q, Q_0 \in \partial\Omega$ and $X \in \Omega \backslash T_{2R}(Q)$ such that $\Delta_s(Q_0) \subset \Delta_R(Q)$, the elliptic measure w^X of L satisfies*

$$w^{A_R(Q)}(\Delta_s(Q_0)) \approx \frac{w^X(\Delta_s(Q_0))}{w^X(\Delta_R(Q))}$$

Proof. Let G denote the corresponding Green's function. Then, Lemma 4.2.3 implies

$$\begin{aligned} w^X(\Delta_R(Q)) &\approx R^{n-2} G(X, A_R(Q)), \\ w^X(\Delta_s(Q_0)) &\approx s^{n-2} G(X, A_s(Q_0)), \\ w^{A_R(Q)}(\Delta_s(Q_0)) &\approx s^{n-2} G(A_R(Q), A_s(Q_0)). \end{aligned}$$

Thus it remains to show that $G(A_R(Q), A_s(Q_0)) \approx \frac{G(X, A_s(Q_0))}{R^{n-2} G(X, A_R(Q))}$ holds for $X \in \Omega \backslash T_{2R}(Q)$.

By the maximum principle, it suffices to show it for $X \in \partial T_{2R}(Q) \cap \Omega$. By (3.8) and (3.9), it holds for $X = A_{2R}(Q)$. The comparison principle and Harnack's principle imply

$$\frac{G(Y, A_s(Q_0))}{G(Y, A_R(Q))} \approx \frac{G(A_{2R}(Q), A_s(Q_0))}{G(A_{2R}(Q), A_R(Q))}$$

for all $Y \in \partial T_{2R}(Q) \cap \Omega$. Thus the proof is complete. $\square$

In Lemma 4.1.2, we have seen that the family of harmonic measures $\{w^X\}_{X\in\Omega}$ is absolutely continuous to each other. For $L \in \mathcal{O}_0$, the doubling property of the elliptic measure implies that the Radon-Nikodym derivative $\frac{\mathrm{d}w^X}{\mathrm{d}w}(Q)$ is given by (see 4.3)

$$K(X, Q) = \lim_{\Delta' \searrow Q} \frac{w^X(\Delta')}{w(\Delta')},$$

in the sense for $\mathrm{d}w$-almost every Q. This means that for $g \in C^0(\partial\Omega)$ and u, the corresponding weak solution, we have

$$u(X) = \int_{\partial\Omega} g(Q) K(X, Q) \,\mathrm{d}w(Q).$$

The Radon-Nikodym derivative K satisfies the following estimate (the proof is from[Ken94], Lemma 1.3.12 and works equally well for $L \in \mathcal{O}_0$):

Lemma 5.3.2. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. For $\Delta_j = \Delta_{2^jR}(Q_0)$ and $A = A_R(Q_0)$ with $2^jR \leq R_0$, we have*

$$\operatorname{ess}\sup_{Q\in\Delta_j\backslash\Delta_{j-1}} K(A, Q) \leq C\frac{2^{-\alpha j}}{w(\Delta_j)}$$

for some $\alpha > 0$ and $1 \leq j \leq \log_2(\frac{R_0}{R})$, and

$$\operatorname{ess}\sup_{Q\in\partial\Omega\backslash\Delta_{R_0}(Q_0)} K(A, Q) \leq C.$$

Proof. Let $\Delta' \subset \Delta_j\backslash\Delta_{j-1}$ be a small surface ball. For $A_j = A_{2^jR}(Q_0)$, Lemma 5.3.1 implies

$$\frac{w(\Delta')}{w(\Delta_j)} \approx w^{A_j}(\Delta'),$$

i.e. $\frac{1}{C}\frac{w(\Delta')}{w(\Delta_j)} \leq w^{A_j}(\Delta') \leq C\frac{w(\Delta')}{w(\Delta_j)}$ for some constant C, which depends as stated in section 2.2 only on $\lambda, \beta, \varepsilon_1, M, \Omega$ and n. From the Hölder continuity up to the boundary, we get

$$w^A(\Delta') \leq C\left(\frac{|A - Q_0|}{2^jR}\right)^\alpha w^{A_j}(\Delta') \leq C2^{-j\alpha}\frac{w(\Delta')}{w(\Delta_j)}.$$

Thus $\frac{w^A(\Delta')}{w(\Delta')} \leq C\frac{2^{-j\alpha}}{w(\Delta_j)}$ for all small Δ' and since $K(X, Q) = \lim_{\Delta'\searrow Q}\frac{w^X(\Delta')}{w(\Delta')}$, the first part of the Lemma is proven.
The second part of the Lemma follows from the comparison principle and Harnack's principle in the interior, since

$$w^A(\Delta') \leq Cw^{A_{R_0}(Q_0)}(\Delta') \approx w(\Delta').$$

$\square$

The proof of Lemma 1.4.2 in [Ken94] gives us the following:

Lemma 5.3.3. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. For a non-negative $g \in C^0(\partial\Omega)$ the non-tangential maximal function of the weak solution u for L with boundary data g satisfies*

$$u^*(P) \approx M_w(g)(P)$$

for all $P \in \partial\Omega$, where $M_w(g)(P) = \sup_{P\in\Delta} \frac{1}{w(\Delta)} \int_\Delta g(Q)\, dw(Q)$ and w is the elliptic measure of L at 0, i.e. a fixed point in the interior of Ω.

Proof. Fix $P \in \partial\Omega$ and choose $X \in \Omega$ such that $X \in \Gamma(P)$. We will show that $u(X) \leq CM_w(g)(P)$. By the interior Harnack principle, we can assume that $X \in (\partial\Omega)_{R_0}$. Let $R = |X-P|$; then by the Lipschitz character of the domain Ω we have $R \approx \delta(X)$. For $\Delta_j = \Delta_{2^jR}(P)$, we have

$$u(X) = \sum_j \int_{\Delta_{j+1}\setminus\Delta_j} K(X,Q)g(Q)\,\mathrm{d}w(Q) + \int_{\Delta_R} K(X,Q)g(Q)\,\mathrm{d}w(Q) = I + II.$$

Lemma 5.3.2 implies $K(X,Q) \leq C\frac{2^{-j\alpha}}{w(\Delta_{j+1})}$ for $Q \in \Delta_{j+1}\setminus\Delta_j$ (for the j's with $2^jR \geq R_0$, we use the fact that there is only a finite number of them and in that case $K(X,Q) \leq C$). Thus

$$I \leq C_\alpha M_w(g)(P).$$

From the definition of K and Lemma 5.3.1, we have $K(X,Q) \approx \frac{1}{w(\Delta_R(P))}$ for $Q \in \Delta_R(P)$. Thus $II \leq CM_w(g)(P)$ and so $u(X) \leq CM_w(g)(P)$ for all $X \in \Gamma(P)$. The same idea shows that

$$u(X) \geq C\frac{1}{w(\Delta_R(P))}\int_{\Delta_R(P)} g(Q)\,\mathrm{d}w(Q).$$

Since this holds for all R and X as long as $X \in \Gamma(P)$ and $|X-P| \approx R$, we get $u^*(P) \approx CM_w(g)(P)$. □

By combining Theorem 5.2.1 with Lemma 5.3.3 we get

Theorem 5.3.4. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Then for $1 < p < \infty$, the following are equivalent:*

- *$(D)_p$ holds*
- *$w \in B_{p'}$*

Moreover, if $(D)_p$ holds, then there exists a $p_0 < p$ such that $(D)_q$ holds for all $q \in (p_0, \infty)$.

With Theorem 5.3.4, we are able to prove the following uniqueness result, which remains unanswered by Theorem 5.1.1.

Theorem 5.3.5. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that $(D)_p$ holds for some $1 < p < \infty$ and that the weak solution u with $u^* \in L^p(\partial\Omega)$ converges non-tangentially almost everywhere to 0. Then $u \equiv 0$.*

Proof. A proof for elliptic operators of the form $L_0 = \mathrm{div}A\nabla$ can be found in [Ken94]. This proof can be adjusted to operators in $\mathcal{O}_0$.
Fix $Z \in \Omega$. We will show that $u(Z) = 0$. Let θ_j be supported in $\Omega_{\frac{1}{2j}}$ with values in $[0,1]$,$\theta_j \equiv 1$ on $\Omega_{\frac{1}{j}}$ and $|\nabla\theta_j| \leq Cj$, where we can assume that j is large. Let $R_j = \Omega_{\frac{1}{2j}}\setminus\Omega_{\frac{1}{j}}$. Then the usage of Green's representation formula (4.2) gives us

$$\begin{aligned}
u(Z) &= u(Z)\theta_j(Z)\\
&= \int_\Omega A\nabla(u\theta_j)(Y)\cdot\nabla_Y G(Z,Y) + G(Z,Y)\,B\nabla(u\theta_j)(Y)\,\mathrm{d}Y\\
&= \int_\Omega A\nabla u(Y)\cdot\nabla_Y G(Z,Y)\,\theta_j(Y)\,\mathrm{d}Y + \int_\Omega G(Z,Y)\,B\nabla u(Y)\,\theta_j(Y)\,\mathrm{d}Y\\
&\quad + \int_\Omega A\nabla\theta_j(Y)\cdot\nabla_Y G(Z,Y)\,u(Y)\,\mathrm{d}Y + \int_\Omega G(Z,Y)\,B\nabla\theta_j(Y)\,u(Y)\,\mathrm{d}Y\\
&= I + II + III + IV.
\end{aligned}$$

For III, we use Caccioppoli's inequality on $G(\cdot, Z)$ and Lemma 4.2.3 to get

$$III \leq \int_{\partial\Omega} (u)^{\frac{1}{j},*}_{\frac{2}{j}}(Q) \left(\sup_{\Delta \ni Q} \frac{w^Z(\Delta)}{|\Delta|} \right).$$

By the assumption, we have $(\sup \frac{w^Z(\Delta)}{|\Delta|}) \in L^{p'}(\partial\Omega)$. Thus $III \to 0$ as $j \to \infty$.
For IV, we use the fact that the vector field B satisfies $|B| \leq Cj$ pointwise on R_j, which allows us to proceed as for III in order to conclude that $IV \to 0$ for $j \to \infty$. For $I + II$, we integrate by parts to see that

$$\begin{aligned} I + II &= \int_\Omega A\nabla u(Y) \cdot \nabla_Y G(Z,Y)\, \theta_j(Y)\, \mathrm{d}Y + \int_\Omega G(Z,Y)\, B\nabla u(Y)\, \theta_j(Y)\, \mathrm{d}Y \\ &= -\int_\Omega G(Z,Y)[(\mathrm{div} A\nabla u(Y))\theta_j(Y) + A\nabla u(Y) \cdot \nabla\theta_j(Y)] \\ &\quad + \int_\Omega G(Z,Y)\, B\nabla u(Y)\, \theta_j(Y)\, \mathrm{d}Y \\ &= -\int G(X,Y)\, A\nabla u(Y) \cdot \nabla\theta_j(Y)\, \mathrm{d}Y. \end{aligned}$$

Thus $I + II \to 0$ for $j \to \infty$ by the same ideas as for III and IV. Therefore, $u(Z) = 0$ and so, $u \equiv 0$. $\square$

Theorem 5.3.4 establishes that the solvability of the Dirichlet problem for boundary data in $L^p(\partial\Omega)$ implies solvability for all $[p, \infty)$ (which can also be deduced by interpolation with the maximum principle) and it implies solvability for indices in $(p_0, p]$ for some $p_0 < p$. If one extends the definition of $(D)_p, 1 < p < \infty$ to $p = \infty$ in the natural way, i.e. by postulating that $||u^*||_{L^\infty(\partial\Omega)} = ||u||_{L^\infty(\Omega)} \leq C||g||_{L^\infty(\partial\Omega)}$ holds for all $g \in C^0(\partial\Omega)$ and corresponding weak solutions u, then one sees by the maximum principle that $(D)_\infty$ holds for all $L \in \mathcal{O}$. But, obviously the maximum principle does not imply $(D)_p$ for some $1 < p < \infty$ in general. Thus, a question one might ask oneself is: is there an appropriate endpoint $(D)_X$ for the $(D)_p$ definition, in the sense that $(D)_p$ for any $1 < p < \infty$ implies $(D)_X$ and $(D)_X$ implies $(D)_q$ for all $q \in (p_0, \infty)$ and some (possibly large) $1 < p_0 < \infty$.
In [DKP10], M. Dindoš, C.E. Kenig and J. Pipher answer this question for elliptic operators of the form $L = \mathrm{div} A\nabla$ with A as in the definition of $\mathcal{O}$.

Definition 5.3.1 ([DKP10], Definition 2.9). *The BMO-Dirichlet problem is solvable for L (denoted by $(D)^L_{BMO}$) if for every continuous boundary data f the corresponding weak solution u satisfies*

$$||\, |\nabla u|^2 \delta(X)\, dX||_{Car} \leq C||f||_{BMO(2)},$$

where $||\, |\nabla u|^2 \delta(X)\, dX||_{Car}$ is the Carleson measure norm of $[|\nabla u|^2 \delta(X)]\, dX$, i.e.

$$||\, |\nabla u|^2 \delta(X)\, dX||^2_{Car} = \sup_{\substack{Q \in \partial\Omega \\ R>0}} \frac{1}{|\Delta_R|} \int_{T_R(Q)} |\nabla u|^2 \delta(X)\, dX$$

and $||f||^2_{BMO(2)} = \sup_\Delta \fint_\Delta |f - f_\Delta|^2\, d\sigma$.

Theorem 5.3.6 ([DKP10], Theorem 2.1). *Assuming that $(D)_{BMO}$ holds, there exists $p_0 < \infty$ such that $(D)_q$ holds for $q \in (p_0, \infty)$. Moreover, $(D)_p$ for any $1 < p < \infty$ implies $(D)_{BMO}$.*

Roughly speaking, the Carleson measure condition on the left hand side of the $(D)_{BMO}$-definition can be interpreted as a square function condition on u, where the square function is defined as $Su(Q) = (\int_{\Gamma(Q)} |\nabla u(X)|^2 \delta^{2-n}(X)\, \mathrm{d}X)^{\frac{1}{2}}$. Since the L^p norms of the square function and the non-tangential maximal function are comparable (see [DJK84] for the precise statement), the $(D)_{\mathrm{BMO}}$ definition is in the spirit of a non-tangential maximal function condition.
In the next chapter, we will deal with the regularity problem and we will answer the same question regarding the appropriate endpoint definition for the regularity problem. This question is already partly answered by C.E. Kenig and J. Pipher in [KP93], where they show that $(R)_p$

implies $(R)_{\mathrm{HS}^1}$, whereas the $(R)_{\mathrm{HS}^1}$ condition is not explicitly defined in [KP93].
We will finish this chapter with an application of the $(D)_{p'}$ and $w \in B_p$ equivalence. Namely, we show that $(D)_p$ is a "boundary thing":

Lemma 5.3.7. *Let $L_0, L_1 \in \mathcal{O}_0$ and $A_0 = A_1, B_0 = B_1$ in $(\partial\Omega)_{\tilde{\beta}}$ for some $\tilde{\beta} > 0$. Then $(D)_{p'}^{L_0}$ implies $(D)_{p'}^{L_1}$.*

Proof. The proof follows from Lemma 4.2.3 and the comparison principle. Let G_i and w_i be the Green's function and the elliptic measure for $L_i, i = 0, 1$. Then for $0 < R < \min\{R_0, \tilde{\beta}\}$, we have

$$\begin{aligned}
\left[\fint_{\Delta_R(Q)} \left(\sup_{s<R} \frac{w_1(\Delta_s(P))}{s^{n-1}}\right)^p \mathrm{d}\sigma(P)\right]^{\frac{1}{p}} &\leq C \left[\fint_{\Delta_R(Q)} \left(\sup_{s<R} \frac{1}{s} \frac{G_1(0, A_s(P))}{G_0(0, A_s(P))} G_0(0, A_s(P))\right)^p\right]^{\frac{1}{p}} \\
\text{(comparison principle)} &\leq C \frac{G_1(0, A_R(Q))}{G_0(0, A_R(Q))} \left[\fint_{\Delta_R(Q)} \left(\frac{w_0(\Delta_s(P))}{s^{n-1}}\right)^p\right]^{\frac{1}{p}} \\
(B_p\text{-condition for } w_0) &\leq C \frac{G_1(0, A_R(Q))}{G_0(0, A_R(Q))} \frac{w_0(\Delta_R(Q))}{R^{n-1}} \approx \frac{w_1(\Delta_R(Q))}{R^{n-1}}
\end{aligned}$$

Thus $w_1 \in B_p$, and therefore $(D)_{p'}^{L_1}$ holds. $\square$

Chapter 6

The Regularity Problem for boundary data in $W^{1,p}(\partial\Omega)$ and HS^1

In Chapter 2, we have seen that for every $f \in C^0(\partial\Omega)$ a unique $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ exists, such that

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{aligned}$$

for $L \in \mathcal{O}$ and Ω a Lipschitz domain. We then further defined the $(D)_p$-condition for $L \in \mathcal{O}_0$, which, under the assumption of the $(D)_p$-condition, allows us to reach the conclusion that for every $f \in L^p(\partial\Omega)$ there exists a unique $u \in W^{1,2}_{loc}(\Omega)$ such that u is a weak solution where u converges non-tangentially almost everywhere to f on $\partial\Omega$ and where one can control the L^p norm of u^* by the L^p norm of f.
If one applies some regularity on the boundary data, namely by assuming that f has tangential derivatives in $L^p(\partial\Omega), 1 < p < \infty$, one is lead to the regularity problem. The regularity problem asks the questions if it is possible to control the "whole" gradient, i.e. the derivative in the tangential and non-tangential direction, of the weak solution u by the tangential derivatives of the boundary data f and if the tangential derivatives of the weak solution converge in some sense, which we will make clear later, to the tangential derivatives of the boundary data. Thus, we seek for a replacement of the $(D)_p$-condition for the regularity problem (which will be denoted by $(R)_p$) such that the above questions can be answered with yes.
The main focus of this chapter is the regularity problem at the endpoint $p = 1$. Considering the duality of H^1 and BMO and the $(D)_{BMO}$ condition from [DKP10], we will extend the definition of the $(R)_p$ condition defined in [KP93] for $1 < p < \infty$ to the endpoint $p = 1$ by defining the $(R)_{\mathrm{HS}^1}$ condition.
In [KP93], Theorem 3.5 it is shown that $(R)_p, 1 < p < \infty$ implies for every $f \in W^{1,p}(\partial\Omega)$ the existence of a weak solution $u \in W^{1,2}_{loc}(\Omega)$ such that u converges non-tangentially to f almost everywhere. With the aid of Theorem 0.1 in [BB10], see Theorem 6.1.4, we are able to derive the same result for the $(R)_{\mathrm{HS}^1}$ case.
In Theorem 5.3 in [KP93] it is proven for $1 < p < \infty$ that $(R)_p$ implies $(R)_{p+\varepsilon}$ for some $\varepsilon > 0$. If one examines the proof of the extrapolation property of the $(R)_p$ condition for $1 < p < \infty$ in [KP93] or [She07], one realises that one gets the extrapolation property of the $(R)_p$ condition as a consequence of the extrapolation property of the $(D^*)_{p'}$ condition (which is a consequence of the theory of weights and the theory of the reverse Hölder class B_p). In [KP93] it is shown that $(R)_p$ implies $(D^*)_{p'}$ for $1 < p < \infty$. The same proof in combination with the $\mathrm{L}\log\mathrm{L}$–characterisation of A_∞ (see Theorem 5.7) allows us to conclude that $(R)_{\mathrm{HS}^1}$ implies $(D^*)_{p'}$ for some $1 < p < \infty$. In that proof it will be important that a Hardy–Sobolev atom a, which will be defined in Definition 6.1.2, does not need to satisfy the cancellation condition $\int a = 0$. The reason why we do not need to impose the cancellation condition on the atoms is the boundedness

of our domain (in [BD09] the domain of interest is unbounded). The characterisation in [BD09] of HS^1 in terms of a maximal function allows us to prove Lemma 6.1.6, which allows us to modify the proof for the implication that $(R)_p$ implies $(R)_{p+\varepsilon}$ for $1 < p < \infty$ and some $\varepsilon > 0$ given in [She07] or [KP93] to show the main result of this thesis, namely that $(R)_{\mathrm{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$. This result in combination with Theorem 5.2 in [KP93], where it is shown that $(R)_p$ implies $(R)_{\mathrm{HS}^1}$, shows that $(R)_{\mathrm{HS}^1}$ is equivalent to $(R)_p$ for some $1 < p < \infty$. This complements the result in [DKP10], which proves that $(D)_{BMO}$ is equivalent to $(D)_p$ for some $1 < p < \infty$.
At the end of this chapter we will look at two more problems: first, we will introduce the $(R)_q$ condition for $q < 1$ in a way that $(R)_q$ and $(D^*)_{p'}$ for $0 < q < 1 < p < \infty$ imply $(R)_p$. Second, for the Neumann problem C.E. Kenig and J. Pipher prove in [KP93] an extrapolation property as well. They show by a duality argument that $(N)_p$ and $(R)_p$ for some $1 < p < \infty$ imply $(N)_{p+\varepsilon}$ for some $\varepsilon > 0$. The duality argument they use seems not to work at the endpoint. With the aid of the method of conjugate solutions we will be able to prove the extrapolation property for the Neumann problem at the endpoint H^1 in two dimensions, namely we show that $(D)_{p'}^{\mathrm{div}\frac{A^T}{\det A}\nabla}$ and $(N)_{H^1}^{\mathrm{div}A\nabla}$ imply $(N)_p^{\mathrm{div}A\nabla}$.

6.1 The Hardy–Sobolev space HS^1

In section 2.1, we defined the smoothness-related function spaces $W^{k,p}, 1 \leq p < \infty$ by first defining the weak derivative in the distributional sense for a local integrable function and then postulating an integrability condition of order p on the weak derivative. In this section, we use a different approach to study the smoothness of functions: concretely, we use a variant of the maximal function to get conclusions about the smoothness of functions.
In [Miy90] and [Cal72], it is shown that a function that has weak derivatives in the Hardy space H^p is equivalent to a maximal function, used by A.P. Calderón and then by A. Miyachi, being bounded on L^p. In [DS84], Theorem 5.3, R. Devore and C. Sharpley show that the maximal function defined by A.P. Calderón is equivalent to a maximal function which we will define now for the case regarding one derivative (see [DS84] (2.2), (4.3), Lemma 2.1, page 36 and page 104 and [BD09]):

Definition 6.1.1. *Let Γ be a domain in $\mathbb{R}^n$. For $0 < q \leq 1$ and $f \in L^q_{loc}(\Gamma)$, we define the maximal function f^b_q by*

$$f^b_q(x) = \sup_{B \ni x} \inf_{c \in \mathbb{R}^n} \frac{1}{r(B)} \left(\fint_B |f - c|^q \right)^{\frac{1}{q}},$$

where the supremum runs over all balls B, which are contained in Γ and contain x. Furthermore, we define the space $\mathcal{C}^q$ as all $f \in L^q_{loc}(\Gamma)$ which fulfil the condition that the norm

$$||f||_{\mathcal{C}^q} = ||f^b_q||_{L^q(\Gamma)} + ||f||_{L^q(\Gamma)}$$

is finite.

For $q = 1$, we see that $f^b_1(x) = \sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \fint_B |f - f_B|$, whereas for $q < 1$, the function f might not be locally integrable, and so f_B might not be defined. To simplify the notation we will write $Nf = f^b_1$, keeping the same notation as in [BD09], i.e.

$$Nf(x) = \sup_{B_r \ni x} \frac{1}{r} \fint_{B_r} |f - f_{B_r}|.$$

In [Haj03] (see (6) in [BD09] as well), it is proven that for $f \in \mathcal{C}^1$, $\frac{s}{s+1} \leq q < 1$, where s is a constant larger than 2 which depends on the doubling property of the underlying metric space, and $q^* = \frac{sq}{s-q}$, the following applies:

$$\left(\fint_{B_r} |f - f_{B_r}|^{q^*}\right)^{\frac{1}{q^*}} \leq Cr \left(\fint_{\tau B_r} |Nf|^q\right)^{\frac{1}{q}} \tag{6.1}$$

for some $\tau > 1$, which is independent of f and r. For $q \in [\frac{s}{s+1}, 1)$, we define

$$\mathcal{M}_q f(x) = \sup_{B \ni x} \left(\fint_B |f|^q\right)^{\frac{1}{q}},$$

where the supremum is taken over all balls containing x.
The result we would like to derive in this section is a version of a result in [BD09]. In [BD09], N. Badr and G. Dafni prove a relationship exists between the Hardy–Sobolev space and the space $\mathcal{C}^1$ on complete Euclidean manifolds M with $\mu(M) = \infty$ and μ a doubling measure. Since we would later like to apply this result on to boundary data in $\partial\Omega$ for Ω a Lipschitz domain, we will not work in such a general setting. Instead our domain will be $\partial\Omega$ for Ω a Lipschitz domain, where the surface measure is the underlying measure. Thus our domain is bounded and has a finite doubling measure. The boundedness of our domain will lead to the fact that Hardy–Sobolev atoms a do not need to satisfy the cancellation condition $\int a = 0$. This will play an important part when we will prove that $(R)_{\mathrm{HS}^1}$ implies $(D^*)_p$ for some $1 < p < \infty$.
We will not write $\partial\Omega$ if there is no confusion possible. In the style of Definition 2.11 and Definition 4.3 in [BD09] and [BB10], we define:

Definition 6.1.2. *For $1 < t \leq \infty$, we say that a function a is a Hardy–Sobolev $(1,t)$-atom, if*

- *a is supported in a ball B*
- $||a||_{L^t} + ||\nabla a||_{L^t} \leq \frac{1}{|B|^{\frac{1}{t'}}}$

We will use the terminology that a is a Hardy–Sobolev $(1,t)$-atom corresponding to the ball B. We define the space HS^1_t as follows: $f \in HS^1_t$, if there exists a family of Hardy–Sobolev $(1,t)$-atoms $\{a_j\}_j$ such that f can be decomposed as

$$f = \sum_j \lambda_j a_j$$

with $\sum_j |\lambda_j| < \infty$. We equip HS^1_t with the norm

$$||f||_{HS^1_t} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all possible decompositions.

The definition of the Hardy–Sobolev space implies $\mathrm{HS}^1_t \subset W^{1,1}$.
If one compares this definition with the Definition 4.1 in [BD09] for non-homogeneous Hardy–Sobolev $(1,t)$-atoms, it becomes clear that we do not impose the cancellation condition $\int a = 0$ on the atoms. This is due to the fact that we do want constant functions to belong to our space. On the other hand our atoms will always satisfy cancellation condition on the level of derivatives:

$$\int_{\partial\Omega} \nabla_T a = 0.$$

If one compares the Definition 6.1.2 with the Definition 2.11 in [BD09] for homogeneous Hardy–Sobolev $(1,t)$-atoms, one can see that N. Badr and G. Dafni impose

$$||a||_{L^1} \leq r(B), \tag{6.2}$$

which automatically holds for our atoms: because for a an atom corresponding to a ball B with $|B| \leq \frac{1}{2}|\partial\Omega|$, we can use Poincaré's inequality and the fact that ∇a is uniformly in L^1. In the

case that $|B| > \frac{1}{2}|\partial\Omega|$, condition (6.2) simplifies to $||a||_{L^1} \leq C$, which obviously holds for any atom.
We will now reproduce the result from [BD09] regarding HS^1_t and $\mathcal{C}^1$ for our setting.

Lemma 6.1.1. *Let a be a Hardy–Sobolev $(1,t)$-atom corresponding to the ball B_0, then*

$$||a||_{\mathcal{C}^1} \leq C_t.$$

Thus $HS^1_t \subset \mathcal{C}^1$ with $||f||_{\mathcal{C}^1} \leq C_t ||f||_{HS^1_t}$.

Proof. We adjust the proof of [BD09], Proposition 4.5, to our setting. In accordance with the Poincaré inequality, we have

$$Na(x) \leq C \sup_{B \ni x} \fint_B |\nabla a| = CM(|\nabla a|)(x).$$

Thus $\int_{2B_0} Na(x) \leq C|B_0|^{\frac{1}{t'}} \left(\int_{2B_0} M(|\nabla a|)^t\right)^{\frac{1}{t}} \leq C_t |B_0|^{\frac{1}{t'}} ||\nabla a||_t \leq C_t$. For $x \in 2^k B_0 \backslash 2^{k-1} B_0$ and $B = B_r(y)$, a ball containing x, we have

$$\begin{aligned}\frac{1}{r}\fint_B |a - a_B| &= \frac{1}{r}\frac{1}{|B|}\Big(\int_{B \cap B_0} |a - a_B| + \int_{B \cap B_0^c} |a_B|\Big) \\ &\leq \frac{3}{r}\frac{1}{|B|}\int_{B_0 \cap B} |a|.\end{aligned}$$

If $B_0 \cap B \neq \emptyset$ then $r > 2^{k-1} r(B_0)$, thus by (6.2) we deduce that $Na(x) \leq C \frac{1}{2^k} \frac{1}{|2^k B_0|}$ for $x \in 2^k B_0 \backslash 2^{k-1} B_0$. Hence $\int_{B_{R_0} \backslash 2B_0} Na(x) \leq C$. Since $||a||_{L^1} \leq C$, the inequality $||a||_{\mathcal{C}^1} \leq C$ is proven for a a Hardy–Sobolev $(1,t)$-atom.
For $f \in \mathrm{HS}^1_t$, take an atomic decomposition $f = \sum \lambda_j a_j$. Then $||f||_{L^1} \leq \sum_j |\lambda_j|$ and $||Nf||_{L^1} \leq C_t \sum_j |\lambda_j|$, thus $||f||_{\mathcal{C}^1} \leq C_t ||f||_{\mathrm{HS}^1_t}$. Since this holds for all decompositions, the proof is complete. ☐

To show the converse, i.e. that $\mathcal{C}^1 \subset \mathrm{HS}^1_t$, we have to construct the Hardy–Sobolev $(1,t)$-atoms, for which we will need the following variant of the Calderón–Zygmund decomposition (see Proposition 4.6 in [BD09]).

Theorem 6.1.2. *Let $f \in \mathcal{C}^1$, $\frac{s}{s+1} \leq q < 1$ and s be as in (6.1). Then for every $\alpha \geq \alpha_0 \approx ||f||_{\mathcal{C}^1}$, one can find balls $\{B_i\}_i \subset \partial\Omega$, functions $b_i \in W^{1,1}$ and $g \in W^{1,\infty}$ such that*

- $f = g + \sum_i b_i$
- $|g| + |\nabla g| \leq C\alpha$ *in the almost everywhere sense*
- $\operatorname{supp} b_i \subset B_i$, $||b_i||_1 \leq Cr_i \alpha |B_i|$, $||b_i||_q + ||\nabla b_i||_q \leq C\alpha |B_i|^{\frac{1}{q}}$
- $\sum_i |B_i| \leq \frac{C}{\alpha} \int (Mf + Nf)$
- $\sum_i \chi_{B_i} \leq C$.

Proof. We adjust the proof for Proposition 4.6 in [BD09]. Consider the open set

$$\Gamma = \{x \in \partial\Omega : \mathcal{M}_q(Mf + Nf)(x) > \alpha\}.$$

Since $f \in \mathcal{C}^1$, it follows from the boundedness of $\mathcal{M}_q$ on L^1 and the Sobolev embedding Theorem that

$$|\Gamma| \leq \frac{C_q}{\alpha} ||Mf + Nf||_{L^1} \leq \frac{C}{\alpha}(||f||_{L^{\tilde{p}}} + ||Nf||_{L^1}) \leq \frac{C}{\alpha} ||f||_{\mathcal{C}^1},$$

where $\tilde{p}$ is the index for the Sobolev embedding Theorem. Thus, for α larger than a fixed constant times $||f||_{\mathcal{C}^1}$, which defines α_0, we have $\Gamma \neq \partial\Omega$.
If $\Gamma = \emptyset$, then we set $f = g$ and are done. Otherwise, we apply a Whitney decomposition (see [Ste70]) to Γ to get balls $\{B_i\}$ with bounded overlap such that

- $\Gamma = \bigcup_i B_i$,
- $r(B_i) \approx \operatorname{dist}(B_i, \Gamma^c)$.

By dilating every ball B_i by a small constant larger than 1, we get new balls with the same properties (and we will still call the new balls B_i to keep the notation simple), for which one can find a partition of unity χ_i of Γ subordinate to the covering B_i such that $\sum_i \chi_i = 1$ on Γ and $|\nabla \chi_i| \leq \frac{C}{r(B_i)}$. By B_i^*, we denote a stretch of B_i such that $B_i^* \cap \Gamma^c \neq \emptyset$. The stretching constant can be chosen uniformly by the properties of the Whitney decomposition. Let $I_x = \{i : x \in B_i\}$, then $\# I_x$ is uniformly bounded by the finite overlap property.
We define b_i by

$$b_i = (f - c_i)\chi_i,$$

where $c_i = \frac{1}{\chi_i(B_i)} \int_{B_i} f\chi_i$ and $\chi_i(B_i) = \int_{B_i} \chi_i$, i.e. c_i is the constant such that the average of b_i is zero. To get a bound for the L_1-norm of b_i we use the fact that $B_i^* \cap \Gamma^c \neq \emptyset$. We have

$$\begin{aligned}
||b_i||_1 &\leq \frac{1}{\chi_i(B_i)} \int_{B_i} \int_{B_i} |f(x) - f(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leq \frac{1}{\chi_i(B_i)} \int_{B_i} \int_{B_i} |f(x) - f_{B_i} + f_{B_i} - f(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leq 2\frac{|B_i|}{\chi_i(B_i)} \int_{B_i} |f - f_{B_i}| \\
&\leq C r_i \left(\fint_{\tau B_i} |Nf|^q \right)^{\frac{1}{q}} |B_i| \qquad (6.1) \\
&\leq C r_i \mathcal{M}_q(Nf)(y_0)|B_i| \quad (y_0 \in \Gamma^c) \\
&\leq C r_i \alpha |B_i|.
\end{aligned}$$

Similarly, for ∇b_i, we get

$$\begin{aligned}
||\nabla b_i||_q &\leq ||\, |(f - c_i)\nabla \chi_i| \,||_q + ||\, |\nabla f|\chi_i||_q \\
&\leq \frac{|B_i|^{\frac{1}{q}-1}}{\chi_i(B_i)} \int_{B_i} \int_{B_i} |f(x) - f(y)||\nabla \chi_i(x)| \,\mathrm{d}y \,\mathrm{d}x + \left(\int_{B_i} |\nabla f|^q \right)^{\frac{1}{q}} \\
&\leq C|B_i|^{\frac{1}{q}} \left(\fint_{B_i} |Nf|^q \right)^{\frac{1}{q}} + |B_i|^{\frac{1}{q}} \left(\fint_{B_i} |Nf|^q \right)^{\frac{1}{q}} \\
&\leq C\alpha |B_i|^{\frac{1}{q}}.
\end{aligned}$$

For the L^q-norm of b_i, we first estimate c_i. For all $y \in B_i$ we have $c_i \leq CMf(y)$, thus with $y_0 \in \Gamma^c \cap B_i^*$, we get $c_i \leq \mathcal{M}_q(Mf)(y_0) \leq C\alpha$. Hence

$$||b_i||_q \leq \left(\int_{B_i} |f - c_i|^q \right)^{\frac{1}{q}} \leq C\alpha |B_i|^{\frac{1}{q}}.$$

Therefore, the $\{b_i\}_i$ satisfy the required conditions. Now set

$$g = f - \sum_i b_i.$$

It remains to be shown that $|g| + |\nabla g| \leq C\alpha$. Since $\sum_i \chi_i \equiv 1$ on Γ, we have

$$\begin{aligned}
\nabla g &= \nabla f - \sum \nabla b_i \\
&= \nabla f - (\sum_i \chi_i)\nabla f - \sum_i (f - c_i)\nabla \chi_i \\
&= \chi_{\Gamma^c} \nabla f - \sum_i (f - c_i)\nabla \chi_i.
\end{aligned}$$

From the definition of Γ, we see that the first term is bounded by $C\alpha$. For the second term we use the fact that $\sum_i \nabla\chi_i \equiv 0$, thus

$$\sum_i (f(x) - c_i)\nabla\chi_i(x) = \sum_{i\in I_x} \left(\fint_{7B_k} f - c_i \right) \nabla\chi_i(x)$$

for any $k \in I_x$. According to the Whitney decomposition, the balls $\{B_i\}_{i\in I_x}$ have comparable radii and therefore, $B_i \subset 7B_k$. Thus, as before, we get

$$|c_i - \fint_{B_{7k}} f| \leq C \fint_{B_i} |f - f_{7B_k}| \leq Cr_k \left(\fint_{(7B_k)^*} |Nf|^q \right)^{\frac{1}{q}} \leq Cr_k\alpha.$$

Hence, the second term is bounded by $C\alpha$ by the finite overlap property of the balls $\{B_i\}$ and so

$$|\nabla g| \leq C\alpha.$$

Since $g = f\chi_{\Gamma^c} + \sum c_i\chi_i$ and $c_i \leq C\alpha$, we see that $g \leq C\alpha$, i.e. the Theorem is proven. $\square$

With the ideas of the proof of Proposition 4.7 in [BD09], we get

Theorem 6.1.3. *Let $f \in \mathcal{C}^1$, $\frac{s}{s+1} \leq q < 1$, s be as in (6.1) and $q^* = \frac{sq}{s-q}(> 1)$. Then, there exists a family of Hardy–Sobolev $(1, q^*)$-atoms $\{a_j\}_j$ such that $f = \sum \lambda_j a_j$ and $\sum_j |\lambda_j| \leq C||f||_{\mathcal{C}^1}$. Thus, $\mathcal{C}^1 \subset HS^1_t$ for $1 < t < q^*$.*

Proof. Let α_0 be as in the proof of Theorem 6.1.2. For every $j \geq j_0$ with j_0 the smallest integer such that $2^{j_0} > \alpha_0$, we apply Theorem 6.1.2 to get

$$f = g^j + \sum_i b_i^j.$$

First, we show that $g^j \to f$ in $W^{1,1}$. Observe that $\int_{B_i^j} |f - c_i^j| \leq r_i^j |B_i^j| Nf(x)$ holds for all $x \in B_i^j$. So

$$\begin{aligned}
||\nabla(g^j - f)||_{L^1} &\leq \sum_i ||\nabla b_i^j||_{L^1} \\
&\leq \sum_i \int_{B_i^j} |f - c_i^j||\nabla\chi_i^j| + \sum_i \int_{B_i^j} |\nabla f| \\
&\leq \sum_i |B_i^j| \left(\fint_{(B_i^j)^*} |Nf|^q \right)^{\frac{1}{q}} + \sum_i \int_{B_i^j} |\nabla f| \\
&\leq C2^j|\Gamma_j| + C\int_{\Gamma_j} |\nabla f|.
\end{aligned}$$

Since $\sum_{j>j_0} 2^j|\Gamma_j| \leq C \int \mathcal{M}_q(Mf + Nf) \leq C||f||_{\mathcal{C}^1}$, we see that the first term tends towards zero and since $\mathcal{M}_q(Mf + Nf)$ is finite almost everywhere, we get that the second term tends towards zero as well. Thus $\nabla g^j \to \nabla f$ in L^1. Since $g^j - f = \sum_i b_i^j$ and all the b_i^j have average zero, we can apply the Poincaré inequality to get $||g^j - f||_{L^1} \leq C||\nabla(g^j - f)||_{L^1} \to 0$. Thus

$$f = \sum_{j\geq j_0} (g^{j+1} - g^j) + g^{j_0}$$

in the $W^{1,1}$ sense. Write l^j for $g^{j+1} - g^j$, then $\operatorname{supp} l^j \subset \Gamma_j$. We can use the partition of unity χ_i^j of Γ_j from the proof of Theorem 6.1.2 to write

$$f = \sum_{j\geq j_0,k} l^j\chi_k^j + g^{j_0}.$$

We have $(|l^j|+|\nabla l^j|)\chi_k^j \le C2^j$. Furthermore,

$$l^j\nabla\chi_k^j = \left(\sum_{\{i:B_k^j\cap B_i^j\neq\emptyset\}}(f-c_i^j)\chi_i^j - \sum_{\{l:B_k^j\cap B_l^{j+1}\neq\emptyset\}}(f-c_l^{j+1})\chi_l^{j+1}\right)\nabla\chi_k^j.$$

Since $\Gamma_{j+1}\subset\Gamma_j$, the balls B_l^{j+1} with $B_k^j\cap B_l^{j+1}\neq\emptyset$ must have radii $r_l^{j+1}\le cr_k^j$. Thus $B_l^{j+1}\subset cB_k^j$ for some positive constant c. One can take c large enough such that $\tau\le c$, where τ is as in (6.1). So we get, by (6.1):

$$\begin{aligned}
(r_k^j)^{q*}\int_{B_k^j}|l^j\nabla\chi_k^j|^{q*} &\le C\int_{B_k^j}\left(\sum_i\chi_{B_i^j}|f-c_i^j|^{q*}+\sum_l\chi_{B_l^{j+1}}|f-c_l^{j+1}|^{q^*}\right)\\
&\le C\sum_{\{i:B_k^j\cap B_i^j\neq\emptyset\}}|B_i^j|(r_i^j)^{q^*}\left(\fint_{\tau B_i^j}|Nf|^q\right)^{\frac{q^*}{q}}\\
&\quad+C\sum_{\{l:B_k^j\cap B_l^{j+1}\neq\emptyset\}}|B_l^{j+1}|(r_l^{j+1})^{q^*}\left(\fint_{\tau B_l^{j+1}}|Nf|^q\right)^{\frac{q^*}{q}}\\
&\le C|B_k^j|(r_k^j2^j)^{q^*}+C(r_l^{j+1}2^{j+1})^{q^*}\sum_{\{l:B_k^j\cap B_l^{j+1}\}}|B_l^{j+1}|\\
&\le C(r_k^j2^j)^{q^*}|B_k^j|.
\end{aligned}$$

Thus $\left(\fint_{B_k^j}|l^j\nabla\chi_k^j|^{q*}\right)^{\frac{1}{q^*}}\le C2^j$ and so $||l^j\chi_k^j||_{W^{1,q*}}\le C2^j|B_k^j|^{\frac{1}{q*}}$. By the choice of α_0 and the definition of g^{j_0} we have

$$||g^{j_0}||_\infty+||\nabla g^{j_0}||_\infty\le C||f||_{\mathcal{C}^1}.$$

Therefore, we can define our atoms a_k^j and a^{j_0} by normalizing $\{(l^j\chi_k^j)\}_{j,k}$ and g^{j_0} appropriately:

$$\begin{aligned}
a_k^j &= C\frac{1}{2^j}\frac{1}{|B_k^j|}l^j\chi_k^j \qquad \text{for } j>j_0\\
\lambda_k^j &= C2^j|B_k^j| \qquad \text{for } j>j_0\\
a^{j_0} &= C\frac{1}{||f||_{\mathcal{C}^1}}g_k^{j_0}\\
\lambda^{j_0} &= C||f||_{\mathcal{C}^1}
\end{aligned}$$

such that $l^j\chi_k^j=\lambda_k^ja_k^j$ and $g_k^{j_0}=\lambda_k^{j_0}a_k^{j_0}$. Then all a_k^j and a^{j_0} are $(1,q^*)$-atoms and

$$\begin{aligned}
\sum_{j\ge j_0}|\lambda_k^j| &\le C||f||_{\mathcal{C}^1}+\sum_{j>j_0,k}2^j|B_k^j|\\
&\le C||f||_{\mathcal{C}^1}+\sum_{j>j_0}2^j|\Gamma_j|\\
&\le C||f||_{\mathcal{C}^1}+\int\mathcal{M}_q(Nf+Mf)\le C||f||_{\mathcal{C}^1}.
\end{aligned}$$

Here we used the Sobolev embedding Theorem in the last inequality. Thus the proof of the Theorem is complete. □

For the application later, we will need a result from [BB10]. Since in our setting, the Poincaré inequality on L^1 holds, Theorem 0.1 in [BB10] tells us the following (the proof is based on another variant of Calderón–Zygmund decomposition, similar to Theorem 6.1.2, which leads to the fact that one can decompose a [BD09] $(1,t)$-atom into [BD09] $(1,\infty)$-atoms):

Theorem 6.1.4. *$HS^1_{t_1} = HS^1_{t_2}$ for all $1 < t_1, t_2 \leq \infty$. The norms are comparable where the implicit constant depends on t_1 and t_2.*

Proof. For the proof, N. Badr and F. Bernicot show that one can decompose every $(1,t)$-atom into $(1,\infty)$-atoms. In their definition, an atom satisfies the cancellation condition $\int a = 0$. The proof in [BB10] still works for our setting, which one can verify directly or by the fact that every Hardy–Sobolev $(1,t)$ atom a can be decomposed into $a = a_{cancel} + a_{bounded}$, where a_{cancel} is a Hardy–Sobolev $(1,t)$-atom that satisfies the cancellation condition and $a_{bounded}$ is a Hardy–Sobolev $(1,\infty)$-atom as in the definition 6.1.2. To see this, let ψ be the cut-off function for the supporting ball B of a. Define

$$a_{bounded} = \left(\frac{|B|}{\int_B \psi} \int a \right) \psi$$

and $a_{cancel} = a - a_{bounded}$. Then $||a_{bounded}||_{L^\infty} \leq |\int a| \leq \frac{1}{|B|}$ and $||\nabla a_{bounded}||_{L^\infty} \leq \frac{1}{r(B)} \leq \frac{C}{|B|}$ since $\partial\Omega$ is a bounded domain. The fact that a_{cancel} is an $(1,t)$-atom which satisfies the cancellation condition is then obvious, since it is the difference of an $(1,t)$-atom and an $(1,\infty)$-atom corresponding to approximately the same ball and $\int a_{cancel} = 0$. $\square$

Thus, we can define $\mathrm{HS}^1 = \mathrm{HS}^1_t$ for any $1 < t \leq \infty$ and we will impose the norm of HS^1_∞ on HS^1. For the application later, the following remark will be important.

Remark 6.1.5. *We have seen that $HS^1 = \mathcal{C}^1$ with equivalent norms. From the construction of the atoms a_j used in the proof, we can see that if $f \in C^0(\partial\Omega)$, then the a_j are in $C^0(\partial\Omega)$.*

In order to keep the notation simple, we assume that we live in $\mathbb{R}^n$ instead of $\partial\Omega$ for the next lemma. This lemma is the main lemma to show that $(R)_{\mathrm{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$.

Lemma 6.1.6. *Fix $0 < R$ and $0 < q \leq 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function for $B_R(0)$, i.e. φ is supported in $B_{2R}(0)$ with values in $[0,1]$, $\varphi \equiv 1$ on $B_R(0)$ and $|\nabla\varphi| \leq \frac{C}{R}$. Assume that $f \in \mathcal{C}^q \cap C^0$ and let $f_{B_{2R}(0)} = \fint_{B_{2R}(0)} f$. Then*

$$||\varphi(f - f_{B_{2R}(0)})||^q_{\mathcal{C}^q} \leq C_q R^n M[M(\nabla f)^q](x) + C_q R^n R^q M(|\nabla f|)(x)^q$$

for any $x \in B_{C_0R}(0)$ and C_0 a constant independent of f.

Proof. First, we claim that for $x \in B_{2R}(0)$, one has $(\varphi[f - f_{B_{2R}(0)}])^b_q(x) \leq M(|\nabla f|)(x)$. For $x \in B_{2R}(0)$, Hölder's inequality implies

$$\begin{aligned}
(\varphi[f - f_{B_{2R}(0)}])^b_q(x) &= \sup_{B \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|B|^{\frac{1}{n}}} \left(\fint_B |\varphi(f - f_{B_{2R}(0)}) - c|^q \right)^{\frac{1}{q}} \\
&\leq \sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \fint_B |\varphi(f - f_{B_{2R}(0)}) - (\varphi[f - f_{B_{2R}(0)}])_B| \\
&\leq C \sup_{B \ni x} \fint_B |\nabla(\varphi[f - f_{B_{2R}(0)}])| \\
&\leq C \sup_{B \ni x} \fint_B |\nabla\varphi||f - f_{B_{2R}(0)}| + \sup_{B \in x} \fint_B \varphi|\nabla f| \\
&\leq \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) > R}} \fint_B \chi_{B_{2R}}|f - f_{B_{2R}(0)}| + \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) \leq R}} \fint_B \chi_{B_{2R}}|f - f_{B_{2R}(0)}| \\
&\quad + M(|\nabla f|)(x) \\
&= I + II + M(|\nabla f|)(x).
\end{aligned}$$

For I, observe that $B \cap B_{2R}(0) \neq \emptyset$ implies $B_{2R}(0) \subset 5B$ and so

$$\begin{aligned} I &\le \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) > R}} \frac{1}{|B|} \int_{B_{2R}} |f - f_{B_{2R}(0)}| \\ &\le \frac{C}{|B_R|} \int_{B_{2R}} |\nabla f| \le M(|\nabla f|)(x). \end{aligned}$$

For II, we use the fact that the uncentered maximal function is dominated by c_n times the centred dyadic maximal function. Hence it is enough to consider the supremum over balls of the form $B_j(x) = B(x, R2^{-j+1}), j \ge 0$. Therefore

$$\begin{aligned} II &\le \frac{C}{R} \sup_{j \ge 0} \fint_{B_j(x)} |f - f_{B_{2R}(0)}| \\ &\le \frac{C}{R} \sup_{j \ge 0} \fint_{B_j(x)} |f - f_{B_j(x)}| + \frac{C}{R} \sum_{j \ge 0} |f_{B_{j+1}(x)} - f_{B_j(x)}| \\ &\le \frac{C}{R} \sup_{j \ge 0} 2^{-j} R \fint_{B_j(x)} |\nabla f| + \frac{C}{R} \sum_{j \ge 0} \fint_{B_{j+1}} |f - f_{B_j(x)}| \\ &\le CM(|\nabla f|)(x) + \frac{C}{R} \sum_{j \ge 0} 2^{-j} R \fint_{B_j(x)} |\nabla f| \\ &\le CM(|\nabla f|)(x) \end{aligned}$$

i.e. the claim is proven. To use the claim, we write

$$||(\varphi[f - f_{B_{2R}(0)}])^b_q||^q_q = \int_{B_{2R}(0)} \left((\varphi[f - f_{B_{2R}(0)}])^b_q\right)^q + \int_{B_{2R}(0)^c} \left((\varphi[f - f_{B_{2R}(0)}])^b_q\right)^q.$$

By the previous claim, the first term is bounded by $CR^n M[M(|\nabla f|)^q](x)$ for any $x \in B_{C_0 R}(0)$. For the second term, we will use the fact that if $x \in B$ and $|x| \approx 2^j R$ then one needs $r(B) \ge C2^j R$ for $B \cap B_{2R} \neq \emptyset$. Thus we have (where $\{|x| \approx 2^j R\}$ denotes the annulus $\{x \in \mathbb{R}^n : 2^j R \le |x| < 2^{j+1} R\}$)

$$\begin{aligned} \int_{B_{2R}(0)^c} \left([\varphi(f - f_{B_{2R}(0)})]^b_q\right)^q &= \sum_{j \ge 1} \int_{\{|x| \approx 2^j R\}} \left[\sup_{B \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|B|^{\frac{1}{n}}} \left(\fint_B |\varphi(f - f_{B_{2R}(0)}) - c|^q \right)^{\frac{1}{q}} \right]^q \mathrm{d}x \\ \{\text{choose } c = 0\} \quad &\le \sum_{j \ge 1} \int_{\{|x| \approx 2^j R\}} \left[\sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \left(\fint_B \chi_{B_{2R}} |f - f_{B_{2R}(0)}|^q \right)^{\frac{1}{q}} \right]^q \\ &\le C \sum_{j \ge 1} \int_{\{|x| \approx 2^j R\}} \left[\frac{1}{2^j R} \left(\frac{1}{(2^j R)^n} \int_{B_{2R}} |f - f_{B_{2R}(0)}|^q \right)^{\frac{1}{q}} \right]^q \mathrm{d}x \\ &\le C \sum_{j \ge 1} (2^j R)^n \frac{1}{(2^j R)^q} \frac{1}{(2^j R)^n} \left(\int_{B_{2R}} |f - f_{B_{2R}(0)}|^q \right) \\ &\le C \sum_{j \ge 1} \frac{1}{(2^j R)^q} \left(\int_{B_{2R}} |f - f_{B_{2R}(0)}| \right)^q |B_{2R}|^{1-q} \\ &\le C_q \left(\int_{B_{2R}} |\nabla f| \right)^q R^{n(1-q)} \\ &\le C_q R^n M(|\nabla f|)(x)^q \end{aligned}$$

for any $x \in B_{C_0 R}(0)$.

To deal with the L^q-norm of $\varphi(f - f_{B_{2R}(0)})$ one can apply Hölder's inequality and Poincaré's inequality to get $||\varphi(f - f_{B_{2R}(0)})||_{L^q} \le CR^n R^q M(|\nabla f|)(x)^q$ for any $x \in B_{C_0 R}$. Thus the proof of the Lemma is complete. $\square$

6.2 The Regularity Problem for boundary data in HS[1]

We have seen that a weak solution u for $L \in \mathcal{O}$ is in $W^{1,2}_{loc}(\Omega)$. In general, ∇u does not exist pointwise, but in the L^2_{loc}sense. Thus the non-tangential maximal might not be the correct tool to define a condition - similarly to the $(D)_p$ condition - that should allow one to relate the gradient of a weak solution to the gradient of the boundary data. For this, we define a variant of the non-tangential maximal function, which is suitable for $L^2_{\text{loc}}(\Omega)$ functions, and which is comparable to the usual non-tangential maximal function for non-negative weak solutions:

$$N_\kappa(h)(Q) = \sup_{X \in \Gamma_\kappa(Q)} \left(\fint_{B_{\frac{\delta(X)}{2}}(X)} |h(Y)|^2 \, \mathrm{d}Y \right)^{\frac{1}{2}} \qquad h \in L^2_{loc}(\Omega). \tag{6.3}$$

We will omit the index κ and write $N(h)$ for $N_\kappa(h)$ if no confusion can arise. Harnack's inequality implies that for u a non-negative weak solution one gets

$$u^*(Q) \approx N(u)(Q).$$

Thus the $(D)_p$-condition is equivalent to $||N(u)||_{L^p(\partial\Omega)} \leq C||f||_{L^p(\partial\Omega)}$ for all $f \in L^p(\partial\Omega) \cap C^0(\partial\Omega)$. Motivated by the $(D)_{BMO}$ condition and the duality of H^1 and BMO, we extend the definition given in [KP93] of the the solvability of the regularity problem for boundary data in $W^{1,p}$ to the endpoint $p = 1$ as follows:

Definition 6.2.1. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}_0$. The regularity problem with boundary data in $W^{1,p}(\partial\Omega), 1 < p < \infty$, is solvable for L (abbreviated $(R)^L_p$), if for every $f \in W^{1,p}(\partial\Omega) \cap C^0(\partial\Omega)$, the weak solution u to the problem*

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial B} & = f \quad \text{on } \partial\Omega \end{cases}$$

verifies

$$||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} \leq C||f||_{W^{1,p}(\partial\Omega)}$$

for a constant C independent of f.
For $p = 1$, the regularity problem with data in $HS^1(\partial\Omega)$ is solvable for L (abbreviated $(R)^L_{HS^1}$) if for every $f \in HS^1(\partial\Omega) \cap C^0(\partial\Omega)$, the corresponding weak solution u to the problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial B} & = f \quad \text{on } \partial\Omega \end{cases}$$

verifies

$$||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} \leq C||f||_{HS^1} \tag{6.4}$$

for a constant C independent of f.
If no confusion can arise, we will omit the index L and write $(R)_p$ and $(R)_{HS^1}$.

For $1 < p < \infty$, the definition of $(R)_p$ is the same as it is introduced by C.E. Kenig and J. Pipher in [KP93] for operators of the form $L_0 = \text{div}A\nabla$. In [KP93], C.E. Kenig and J. Pipher prove the following - the proof works equally well for $L \in \mathcal{O}_0$ and Ω a Lipschitz domain:

Theorem 6.2.1 (Theorem 3.1 and Theorem 3.5 in [KP93])**.** *Let $L_0 = divA\nabla$ for A as in the definition of $\mathcal{O}$ and symmetric and Ω be a star-shaped Lipschitz domain. Assuming that $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution for L_0 in Ω with $u \in L^p(\Omega)$ and $N(\nabla u) \in L^p(\partial\Omega)$ for some $1 < p < \infty$, u converges non-tangentially almost everywhere to a function f with $f \in W^{1,p}(\partial\Omega)$. In addition*

$$(\nabla_T u)_r(Q) = \fint_{B_{\frac{r}{2}}(A_r(Q))} \nabla u(X) \cdot \vec{T}(Q) \, dX$$

converges weakly in $L^p(\partial\Omega)$ to $\nabla_T f$ for $r \to 0$. If $f = 0$ almost everywhere, then $u \equiv 0$. Supposing further that $(R)_p$ holds, then there exists a unique $u \in L^p(\Omega)$ with $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} \le C||f||_{W^{1,p}(\partial\Omega)}$ for every $f \in W^{1,p}(\partial\Omega)$ and $Lu = 0$ in Ω such that u converges non-tangentially almost everywhere to f and $(\nabla_T u)_r \to \nabla_T f$ in $L^p(\partial\Omega)$ weakly.

In the rest of this section, we proof the equivalent of Theorem 6.2.1 for the $(R)_{\mathrm{HS}^1}$ condition. The proofs and ideas used in [KP93] for Theorem 3.1 and Theorem 3.5 work almost equally well to get similar results for the $(R)_{\mathrm{HS}^1}$ case with Ω a Lipschitz domain and $L \in \mathcal{O}_0$.

Theorem 6.2.2. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}_0$. Assume that $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution and that $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} < \infty$. Then,*

- *u converges non-tangentially almost everywhere to a function f with $f \in W^{1,1}(\partial\Omega)$.*
- *If $f = 0$ almost everywhere, $u \equiv 0$.*
- *There exists a sequence $r_j \to 0$ for $j \to \infty$ such that $(\nabla_T u)_{r_j}$ converges in the weak* topology of $(L^\infty(\partial\Omega))^*$ to $\nabla_T f$.*

Proof. We follow the proof of Theorem 3.1 in [KP93]. The fact that $N(\nabla u) \in L^1(\partial\Omega)$ implies that $N(\nabla u) < \infty$ almost everywhere, and since Ω is a Lipschitz domain, the interior normal $\vec{N}(Q)$ of $\partial\Omega$ at Q exists almost everywhere. Choose a Q such that both almost everywhere conditions are valid and let $X, Y \in \Gamma(Q) \cap B_\varepsilon(Q)$ for some small $\varepsilon > 0$. We claim that $|u(X) - u(Y)| \le C\varepsilon N(\nabla u)(Q)$, which implies that $u(X_n)$ for $\Gamma(Q) \ni X_n \to Q$ is a Cauchy sequence and therefore that u has a non-tangential limit at Q, which we will denote by $f(Q)$. Choose $0 < r_1, r_2 < \varepsilon$ such that $X_1 = Q + r_1\vec{N}(Q), Y_1 = Q + r_2\vec{N}(Q)$ with $|X - X_1| = \min_{r>0} |X - (Q + r\vec{N}(Q))|, |Y - Y_1| = \min_{r>0} |Y - (Q - r\vec{N}(Q))|$. Without losing generality, we can assume that there exist balls[1] B_1 and B_2 which are centred at X_1 and Y_1 with radii $r(B_1) \approx |X - Q|$ and $r(B_2) \approx |Y - Q|$ such that $X \in B_1, Y \in B_2$ and $2B_1 \subset \Omega, 2B_2 \subset \Omega$. We have

$$|u(X) - u(Y)| \le |u(X) - u(X_1)| + |u(X_1) - u(Y_1)| + |u(Y_1) - u(Y)|.$$

By the equivalent of Theorem 2.2.7 in the inside (see Theorem 8.17 in [GT01]) and the Poincaré inequality, we get

$$|u(X) - u(X_1)| \le Cr(B_1)\left(\fint_{B_1} |\nabla u|^2\right)^{\frac{1}{2}} \le Cr(B_1)N(\nabla u)(Q),$$
$$|u(Y_1) - u(Y)| \le Cr(B_2)\left(\fint_{B_2} |\nabla u|^2\right)^{\frac{1}{2}} \le Cr(B_2)N(\nabla u)(Q).$$

Without loss of generality, we can assume that $r_1 \ge r_2$. Choose a sequence $\{t_j\}_{j=1}^N$ with $t_1 = r_1, t_N = r_2$ and $t_j \approx 2^{-j}r_1$ and call the points $Q + t_j\vec{N}(Q) = Z_j$. For Z_j and $Z_{j+1}, j = 1, \cdots, N-1$, we get the corresponding estimate as above for $u(X) - u(X_1)$ and $u(Y) - u(Y_1)$. Hence

$$\begin{aligned}|u(X_1) - u(Y_1)| &\le \sum_{j=1}^{N-1} |u(Z_j) - u(Z_{j+1})| \\ &\le \sum_{j\ge 1} 2^{-j}r_1 N(\nabla u)(Q) \le C\varepsilon N(\nabla u)(Q).\end{aligned}$$

[1] If those balls do not exist consider

$$|u(X) - u(X_1)| \le |u(X) - u(\tilde{X}_1)| + |u(\tilde{X}_1) - u(\tilde{X}_2)| + \cdots + |u(\tilde{X}_K) - u(X_1)|,$$

where $\tilde{X}_k, k = 1, \cdots K$ is a finite sequence of points on the line from X to X_1 such that $|X - \tilde{X}_1|, |X_1 - \tilde{X}_K| \le \frac{\delta(X)}{2}$ and $|\tilde{X}_k - \tilde{X}_{k+1}| \approx \frac{\delta(X)}{2}$ for $k = 2, \cdots, K$. The constant K is finite and depends on the aperture of the non-tangential maximal function. One can then apply the same methods, which are used in the proof, to each term. Since the number of terms is uniformly bounded by a constant depending on the aperture, the proof works equally well. The same applies to $|u(Y) - u(Y_1)|$.

Thus we have proven that for $X, Y \in \Gamma(Q) \cap B_\varepsilon(Q)$ one has

$$|u(X) - u(Y)| \leq C\varepsilon N(\nabla u)(Q) \tag{6.5}$$

for almost every $Q \in \partial\Omega$, which completes the claim and means that u converges non-tangentially almost everywhere to a function f.
The inequality (6.5) implies that $u^* \in L^1(\partial\Omega)$ since $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} < \infty$ and so $f \in L^1(\partial\Omega)$.
In the $1 < p < \infty$ case of Theorem 6.2.1, C.E. Kenig and J. Pipher proceed by showing that

$$N_{\text{Calderón}}(f)(Q) = \sup_{t>0} \frac{1}{t} \int_{\Delta_t(Q)} |f(P) - f(Q)| \, \mathrm{d}P$$

is in $L^p(\partial\Omega)$, where $N_{\text{Calderón}}$ is the maximal function used by Calderón [Cal72] and is pointwise equivalent to $Nf(Q)$ by the results in [DS84] (see Definition 6.1.1). For this, C.E. Kenig and J. Pipher show that $N_{\text{Calderón}}$ is pointwise dominated by $M(N(\nabla u))(Q)$, which shows that $f \in W^{1,p}(\partial\Omega)$ by the boundedness of the Hardy–Littlewood maximal function on L^p for $1 < p < \infty$.
In the case that $p = 1$, we claim the following two statements

- $(\nabla_T u)_r \to \nabla_T f$ in the sense of distributions
- $||(\nabla_T u)_r||_{L^1(\partial\Omega)} \lesssim ||N(\nabla u)||_{L^1(\partial\Omega)}$

These two facts imply $f \in W^{1,1}(\partial\Omega)$ and the statement about the convergence of $(\nabla_T u)_{r_j}$, because the unit ball of any weak* topology is compact. Therefore, there exists a subsequence r_j and a measure $\mu \in (L^\infty(\partial\Omega))^*$ such that $(\nabla_T u)_{r_j} \rightharpoonup \mu$ in the weak* topology of $(L^\infty(\partial\Omega))^*$, i.e. for each $\psi \in L^\infty(\partial\Omega)$ we have

$$\lim_{r_j \to 1} \int_{\partial\Omega} (\nabla_T u)_{r_j}(Q) \, \psi(Q) \, \mathrm{d}\sigma(Q) = \int_{\partial\Omega} \psi(Q) \, \mathrm{d}\mu(Q).$$

This implies that $(\nabla_T u)_{r_j}$, seen as a measure, converges set-wise to μ and μ is finitely valued. The measures $(\nabla_T u)_{r_j} \, \mathrm{d}\sigma$ are absolutely continuous with respect to the surface measure and therefore, by the Vitali-Hahn-Soks Theorem (see [Doo94], p.155 for example), μ is absolutely continuous with respect to the surface measure. Hence, there exists $g \in L^1(\partial\Omega)$ such that $\mu = g \, \mathrm{d}\sigma$. Due to the uniqueness of limits in the sense of distributions, we see that $g = \nabla_T f$.
To see that $(\nabla_T u)_r \to \nabla_T f$ in the sense of distributions, which is the statement of the first claim, let $\theta \in C^\infty(\partial\Omega)$. In order to illustrate the idea of the following argument, assume that $\Omega = \mathbb{R}^2_+, \theta$ is in $C_0^\infty(\mathbb{R})$ and $u^* \in L^1(\mathbb{R})$ with u converging non-tangentially to f almost everywhere. Let ∇_y denote the tangential derivative, then

$$\begin{aligned}
\int_{\mathbb{R}} (\nabla_y u)_r(x)\theta(x) \, \mathrm{d}x &= \int_{\mathbb{R}} \fint_{B_{\frac{r}{2}}(x,r)} \nabla_y u(y,s) \, \mathrm{d}y \, \mathrm{d}s \, \theta(x) \, \mathrm{d}x \\
&= -\pi \int_{\mathbb{R}} \fint_{[\frac{r}{2}, \frac{3r}{2}]} u(y,s) \, \nabla_y \left(\frac{1}{r} \int_{[y-\gamma_r(s), y+\gamma_r(s)]} \theta(x) \, \mathrm{d}x \right) \mathrm{d}s \, \mathrm{d}y,
\end{aligned}$$

where $\gamma_r(s) = \sqrt{\frac{r^2}{4} - (r-s)^2}$ for $s \in [\frac{r}{2}, \frac{3r}{2}]$. Then $\nabla_y \left(\frac{1}{r} \int_{[y-\gamma_r(s), y+\gamma_r(s)]} \theta(x) \, \mathrm{d}x \right) \leq \mathrm{Lip}(\theta)$, where $\mathrm{Lip}(\theta)$ is the Lipschitz constant of θ, i.e. $\mathrm{Lip}(\theta) = \sup_{x,y \in \mathbb{R}} \frac{|\theta(x) - \theta(y)|}{|x-y|}$. In addition, $|\fint_{[\frac{r}{2}, \frac{3r}{2}]} u(y,s) \, \mathrm{d}s \leq u^*(y)$. Hence the dominated convergence Theorem implies (since $u^* \in L^1(\mathbb{R})$) that

$$\lim_{r \to 0} \int_{\mathbb{R}} (\nabla_y u)_r(x,r)\theta(x) \, \mathrm{d}x = -\int_{\mathbb{R}} f(y) \nabla_y \theta(y) \, \mathrm{d}y.$$

Thus applying these thoughts, i.e interchanging the order of integration and integrating by parts, allows us to conclude $\lim_{r \to 1} \int_{(\partial\Omega)} (\nabla_T u)_r \theta = -\int_{(\partial\Omega)} f \nabla_T \theta = \int_{(\partial B)} \nabla_T f \theta$, since $u \to f$ non-tangentially almost everywhere and $u^* \in L^1(\partial\Omega)$.

For the statement of the second claim we observe that since $(\nabla_T u)_r(Q) \leq N(\nabla u)(Q)$, we see that $||(\nabla_T u)_r||_{L^1(\partial\Omega)} \leq C||N(\nabla u)||_{L^1(\partial\Omega)}$, which completes the proof of the two claimed statements.
So far, we have not made use of the fact that u is a weak solution and that $L \in \mathcal{O}_0$. We will need these conditions to prove that $f = 0$ almost everywhere implies $u \equiv 0$. The proof is very similar to the proof regarding the $(D)_p$ case, whereas we do not need to assume that the $(R)_p$ condition holds in the $(R)_p$ case.
Fix $Z \in \Omega$. Let θ_j satisfy $\theta_j \equiv 1$ on $\Omega_{\frac{1}{j}}, \operatorname{supp}\theta_j \subset \Omega_{\frac{1}{2j}}$ and $|\nabla\theta_j| \leq Cj$. We assume that j is large enough so that we can see Z as a fixed constant away from $(\partial\Omega)_{\frac{1}{j}}$. The Green's representation formula (4.2) implies

$$\begin{aligned}
u(Z) &= u(Z)\theta_j(Z) \\
&= \int_\Omega A\nabla(u\theta_j)(Y)\cdot\nabla_Y G(Z,Y) + G(Z,Y)\, B\nabla(u\theta_j)(Y)\,\mathrm{d}Y \\
&= \int_\Omega A\nabla u(Y)\cdot\nabla_Y G(Z,Y)\,\theta_j(Y)\,\mathrm{d}Y + \int_\Omega G(Z,Y)\, B\nabla u(Y)\,\theta_j(Y)\,\mathrm{d}Y \\
&+ \int_\Omega A\nabla\theta_j(Y)\cdot\nabla_Y G(Z,Y)\,u(Y)\,\mathrm{d}Y + \int_\Omega G(Z,Y)\, B\nabla\theta_j(Y)\,u(Y)\,\mathrm{d}Y \\
&= I + II + III + IV.
\end{aligned}$$

For B_j, a ball of radius $\frac{1}{10j}$ and centred at a point with distance $\frac{1}{j}$ to the boundary, we get (by the Cacciopoli inequality and the Hölder continuity up to the boundary)

$$\fint_{B_j} |\nabla_Y G(Z,Y)|\,\mathrm{d}Y \leq \left(\fint_{B_j} |\nabla_Y G(Z,Y)|^2\,\mathrm{d}Y\right)^{\frac{1}{2}} \leq C\frac{j}{j^\alpha}, \tag{6.6}$$

$$\fint_{B_j} G(Z,Y)\,\mathrm{d}Y \leq C\frac{1}{j^\alpha}. \tag{6.7}$$

Since the vector field B satisfies $|B(Y)|\delta(Y) \leq C$, we have for $Y \in \operatorname{supp}\nabla\theta_j$ and Q such that $Y \in \Gamma(Q)$

$$\begin{aligned}
|B(Y)| &\leq Cj \\
u(Y) &\leq C\frac{1}{j}N(\nabla u)(Q).
\end{aligned}$$

Let $R_j = \operatorname{supp}\nabla\theta_j$. Choose essentially disjoint balls B_j^k like B_j as above, such that $R_j = \bigcup_{k=1}^N B_j^k$ with $N \approx 1$. By shrinking the balls B_k^j by a fixed constant, which depends on the Lipschitz domain but is independent of j, we can assume that there exist essentially disjoint surface balls Δ_j^k of diameter comparable to $\frac{1}{j}$ such that $B_j^k \subset \Gamma(Q)$ for every $Q \in \Delta_j^k$. So for III we get

$$\begin{aligned}
III &\leq Cj\int_{R_j} |\nabla_Y G(Z,Y)||u(Y)|\,\mathrm{d}Y \\
&\leq C\sum_k \left(\frac{1}{j}\right)^{n-1}\left(\fint_{B_j^k} |\nabla_Y G(Z,Y)|^2\right)^{\frac{1}{2}}\left(\fint_{B_j^k} |u(Y)|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and therefore

$$\begin{aligned}
III &\leq Cj\sum_k \left(\frac{1}{j}\right)^{n-1}\frac{1}{j^\alpha}\fint_{\Delta_j^k} N(\nabla u) \\
&\leq C\frac{1}{j^\alpha}\sum_k \int_{\Delta_j^k} N(\nabla u) \leq C\frac{1}{j^\alpha}\int_{\partial\Omega} N(\nabla u).
\end{aligned}$$

Similar thoughts apply to IV: namely we have by (6.7) that $\sup_{Y\in B_j} B(Y) \fint_{B_j} G(Z,Y)\,\mathrm{d}Y \leq C\frac{j}{j^\alpha}$ which is the same bound as for $\fint_{B_j} G(Z,Y)$ in (6.6). Hence the estimates for III work equally well for IV.
For $I+II$, one integrates by parts to get

$$\begin{aligned} I+II &= \int_\Omega A\nabla u(Y)\cdot\nabla_Y G(Z,Y)\,\theta_j(Y)\,\mathrm{d}Y + \int_\Omega G(Z,Y)\,B\nabla u(Y)\,\theta_j(Y)\,\mathrm{d}Y \\ &= -\int_\Omega G(Z,Y)[(\mathrm{div}A\nabla u(Y))\theta_j(Y) + A\nabla u(Y)\cdot\nabla\theta_j(Y)] \\ &\quad + \int_\Omega G(Z,Y)\,B\nabla u(Y)\,\theta_j(Y)\,\mathrm{d}Y \\ &= -\int G(X,Y)\,A\nabla u(Y)\cdot\nabla\theta_j(Y)\,\mathrm{d}Y, \end{aligned}$$

since $\mathrm{div}A\nabla u - B\nabla u = 0$. Thus by the Hölder continuity up to the boundary

$$\begin{aligned} (I+II) &\leq Cj\int_{R_j} G(Z,Y)|\nabla u(Y)|\,\mathrm{d}Y \\ &\leq C\frac{j}{j^\alpha}\int_{R_j}|\nabla u(Y)|\,\mathrm{d}Y \leq C\frac{1}{j^\alpha}\int_{\partial\Omega} N(\nabla u)(Q)\,\mathrm{d}Q. \end{aligned}$$

So $u(Z) \leq \frac{1}{j^\alpha}\int_{\partial\Omega} N(\nabla u)$, which implies that $u(Z)=0$, and therefore $u\equiv 0$ since $Z\in\Omega$ was arbitrary. $\square$

The following lemma shows that it suffices to assume the $(R)_{\mathrm{HS}^1}$ condition for smooth Hardy–Sobolev atoms (compare Remark 3.4 in [KP93] for the $(R)_p$ case), where we call a Hardy–Sobolev atom a a smooth Hardy–Sobolev atom, if a is a Hardy–Sobolev atom and is smooth:

Lemma 6.2.3. *Assuming that the estimate (6.4) holds for smooth Hardy–Sobolev atoms, then the $(R)_{HS^1}$ condition holds.*

Proof. We first claim that if the estimate (6.4) holds for all continuous Hardy–Sobolev atoms then $(R)_{\mathrm{HS}^1}$ holds. Indeed, let $f\in \mathrm{HS}^1\cap C^0(\partial\Omega)$. Then according to the Remark 6.1.5, there exist continuous atoms a_j and scalars λ_j such that $f=\sum\lambda_j a_j$. Thus if u is the solution for f, and u_j for a_j, we have

$$\begin{aligned} ||N(\nabla u)||_{L^1(\partial\Omega)} &\leq \sum_j|\lambda_j|\;||N(\nabla u_j)||_{L^1(\partial\Omega)} \leq C\sum_j|\lambda_j|, \\ ||u||_{L^1(\Omega)} &\leq \sum_j|\lambda_j|\;||u_j||_{L^1(\Omega)} \leq C\sum_j|\lambda_j|. \end{aligned}$$

Since this holds for all decompositions, we get $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} \leq C||f||_{\mathrm{HS}^1}$ and so the claim holds. Hence it is enough to prove (6.4) for continuous Hardy–Sobolev atoms a under the assumption that (6.4) holds for smooth Hardy–Sobolev atoms.
Every continuous Hardy–Sobolev atom a can be uniformly approximated in HS^1 by smooth Hardy–Sobolev atoms a_j (use mollifiers). Let u_j be the weak solution for the smooth Hardy–Sobolev atom a_j and u the solution for a. The maximum principle implies that u_j converges uniformly to u on $\bar\Omega$, hence $||u||_{L^1(\Omega)} \leq \lim_j ||u_j||_{L^1(\Omega)} \leq C\lim_j ||a_j||_{\mathrm{HS}^1} \leq C||a||_{\mathrm{HS}^1}$. Let

$$N_\varepsilon(h)(Q) = \sup_{\substack{X\in\Gamma(Q)\\ \delta(X)\geq\varepsilon}} \left(\fint_{B(X,\delta(X)/2)}|h|^2\right)^{\frac{1}{2}}$$

be the truncated below maximal function. Cacciopoli's inequality and the uniform convergence of u_j to u imply $N_\varepsilon(\nabla u_j - \nabla u) \to 0$ uniformly with respect to Q. Therefore,

$$\int_{\partial\Omega} N_\varepsilon(\nabla u) \leq \lim_{j\to\infty} \int_{\partial\Omega} N_\varepsilon(\nabla u_j) \leq C \lim_j ||a_j||_{\mathrm{HS}^1} \leq C||a||_{\mathrm{HS}^1}.$$

Since N_ε increases to N, the monotone convergence theorem completes the proof. $\square$

Recall that when we defined the $(R)_{\mathrm{HS}^1}$ solvability we only did it for data in $\mathrm{HS}^1 \cap C^0(\partial\Omega)$. The following theorem shows that this is sufficient and that this implies existence of a unique solution for any data in HS^1.

Theorem 6.2.4. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that $(R)_{HS^1}$ holds. Given $f \in HS^1$, there exists a unique $u \in L^1(\Omega)$ with $N(\nabla u) \in L^1(\partial\Omega)$ such that $Lu = 0$ in Ω and that u converges non-tangentially almost everywhere to f. Additionally, $(\nabla_T u)_{r_j}$ converges in the weak* topology of $(L^\infty(\Omega))^*$ to $\nabla_T f$.*

Proof. We use, where it is possible, the ideas from the proof of Theorem 3.5 in [KP93]. By Theorem 0.1 in [BB10] we have seen that the norms of $\mathrm{HS}^1_{t_1}$ and $\mathrm{HS}^1_{t_2}$ for $1 < t_1, t_2 \leq \infty$ are equivalent, thus every $(1,\infty)$-atom can be approximated by smooth $(1,\infty)$-atoms in HS^1. Let $f = \sum_j \lambda_j a_j$, then choose smooth $(1,\infty)$-atoms a_j^N with $||a_j^N - a_j||_{\mathrm{HS}^1} \leq \varepsilon \frac{1}{2^j} \frac{1}{\sum_j |\lambda_j|}$. Now choose N such that $\sum_{j>N}^\infty |\lambda_j| \leq \varepsilon$. Then for $f^N = \sum_{j=1}^N \lambda_j a_j^N$ we have $||f - f_N||_{\mathrm{HS}^1} \leq \sum_{j=1}^N |\lambda_j| ||a_j^N - a_j||_{\mathrm{HS}^1} + \sum_{j>N}^\infty |\lambda_j| \leq 2\varepsilon$, i.e. $f_N \to f$ in HS^1 with f_N smooth.
It follows that we can choose $f_j \in \mathrm{HS}^1 \cap C^\infty(\partial\Omega)$ converging in $\mathrm{HS}^1$to $f \in \mathrm{HS}^1$. Let u_j be the weak solution for the smooth boundary data f_j. Then

$$||N(\nabla(u_j - u_k))||_{L^1(\partial B)} + ||u_j - u_k||_{L^1(B)} \to 0,$$

and so $\{u_j\}_j$ is a Cauchy sequence in $L^1(\Omega)$. Thus, there exists a $u \in L^1(B)$ such that $u_j \to u$ in $L^1(B)$. Using Caccioppoli's inequality in the interior, we see that for any compact $K \subset \Omega$, one has

$$||u_j - u_k||_{W^{1,2}(K)} \leq C_K ||u_j - u_k||_{L^1(\Omega)} \to 0.$$

The uniqueness of limits implies $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Since the vector field $B \in L^\infty(K)$ for any compact $K \subset \Omega$, one gets that u is a weak solution. Furthermore,

$$\begin{aligned} ||u||_{L^1(\Omega)} &= \lim_{j\to\infty} ||u_j||_{L^1(\Omega)} \leq C \lim_j ||f_j||_{\mathrm{HS}^1} \leq C||f||_{\mathrm{HS}^1} \\ ||u - u_j||_{L^1(\Omega)} &\leq C||f - f_j||_{\mathrm{HS}^1}. \end{aligned}$$

By using the same N_ε-idea as before, we get

$$\begin{aligned} &||N(\nabla u)||_{L^1(\partial\Omega)} \leq C||f||_{\mathrm{HS}^1} \\ &||N(\nabla(u - u_j))||_{L^1(\partial\Omega)} \leq C||f - f_j||_{\mathrm{HS}^1}, \end{aligned}$$

Hence Theorem 6.2.2 implies that u has a non-tangential limit almost everywhere, which we will denote by $u|_{\partial\Omega}$. It remains to be verified that $u|_{\partial B} = f$ almost everywhere. We know by (6.5) that

$$(u_j - u)^*(Q) \leq CN(\nabla(u - u_j))(Q) + C||u_j - u||_{L^1(\Omega)}.$$

Therefore,

$$\begin{aligned} |\{|f - u|_{\partial\Omega}| > \alpha\}| &\leq |\{|f - f_j| > \frac{\alpha}{3}\}| + |\{|f_j - u_j|_{\partial\Omega}| > \frac{\alpha}{3}\}| + |\{|u_j|_{\partial\Omega} - u|_{\partial\Omega}| > \frac{\alpha}{3}\}| \\ &\leq \frac{C}{\alpha} ||f - f_j||_{L^1(\partial\Omega)} + |\{(u_j - u)^* \geq \alpha\}| \\ &\leq \frac{C}{\alpha} \left(||f - f_j||_{L^1(\partial\Omega)} + ||N(\nabla(u_j - \nabla u))||_{L^1(\partial\Omega)} + ||u_j - u||_{L^1(\Omega)}\right) \\ &\leq \frac{C}{\alpha} ||f - f_j||_{\mathrm{HS}^1}, \end{aligned}$$

which implies the non-tangential almost everywhere convergence. Uniqueness and the stated $(\nabla_T u)_{r_j}$ convergence follow from Theorem 6.2.2, which completes the proof. $\square$

6.3 Consequences of the $(R)_{\mathbf{HS}^1}$-condition

In [KP93], C.E. Kenig and J. Pipher prove an interpolation and extrapolation property of the $(R)_p$ condition for operators of the form $L_0 = \mathrm{div} A\nabla$ for A, as in the definition of $\mathcal{O}$ and symmetric. Concretely, they prove that for $1 < p < \infty$ $(R)_p$ implies $(D)_{p'}$ and that $(R)_p$ implies $(R)_q$ for $1 \leq q < p+\varepsilon$ and some $\varepsilon > 0$, where $(R)_1$ is to be understood as $(R)_{\mathrm{HS}^1}$. The range $1 < q \leq p$ is achieved by first showing that $(R)_{\mathrm{HS}^1}$ holds under the assumption of the $(R)_p$ condition[2] (the $(R)_{\mathrm{HS}^1}$ condition is not defined formally in [KP93]), for which they use a localization Theorem which is valid under the assumption of the $(R)_p$ condition, and secondly by interpolating between $(R)_{\mathrm{HS}^1}$ and $(R)_p$ they get the range $(1, p]$. C.E. Kenig and J. Pipher use the same localization Theorem in combination with a real variable technique to prove the existence of an $\varepsilon > 0$ such that $(R)_p$ for $1 < p < \infty$ implies $(R)_q$ for $q \in [p, p+\varepsilon)$.
In [She07], Z. Shen uses a reverse Hölder inequality for $N(\nabla u)$ to show that for the same sort of elliptic operators as in [KP93] either $(D)_{p'}$ implies $(R)_p$ or $(R)_q$ is not solvable for any $1 < q < \infty$.
In this section, we will prove several consequences of the $(R)_{\mathrm{HS}^1}$ condition. The first consequence will be that $(R)_{\mathrm{HS}^1}$ implies $(D^*)_{p'}$ for some $1 < p < \infty$, which extends the result in [KP93], where it is shown that $(R)_p$ implies $(D)_{p'}$ for $1 < p < \infty$ and $L_0 = \mathrm{div} A\nabla$ with A symmetric. In addition, we will present a new proof for $(R)_p$ implying $(R)_{\mathrm{HS}^1}$ without the localization Theorem by combining ideas in [She07] and [KP93]. Furthermore, we will prove the main Theorem of this thesis, namely that $(R)_{\mathrm{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$. This result allows us to include the endpoint in Shen's result (see Theorem 6.3.12). As a consequence of the main result and the new proof that $(R)_p$ implies $(R)_{\mathrm{HS}^1}$ without using the localisation Theorem we will get Corollary 6.3.13, which reduces the requirements for a possible proof of the conjecture that $(D^*)_{p'}$ implies $(R)_p$ for $1 < p < \infty$.

6.3.1 $(R)_{\mathbf{HS}^1}$ implies $(D^*)_{p'}$ for some $1 < p < \infty$

Let us recall a variant of the non-tangential maximal function from [KP93]. For any $h : \Omega \to \mathbb{R}, Q \in \partial\Omega$ we consider $S_{\varepsilon,R}(Q) = T_R(Q) \cap (\partial\Omega)_{\varepsilon R}$ and define

$$N^\varepsilon(h)(Q) = \sup_{X \in \Gamma(Q)} \left(\fint_{T_{\delta(X)}(\hat{X}) \setminus S_{\varepsilon,\delta(X)}(\hat{X})} |\nabla h|^2 \, \mathrm{d}Z \right)^{\frac{1}{2}}.$$

Lemma 6.3.1. *For all $0 < p < \infty$, there exists C_1, C_2 depending only on ε, p and Ω such that*

$$C_1 ||N^\varepsilon(h)||_{L^p(\partial\Omega)} \leq ||N(h)||_{L^p(\partial\Omega)} \leq C_2 ||N^\varepsilon(h)||_{L^p(\partial\Omega)}.$$

Proof. As it is stated in [KP93], the proof can be found in [FS72], Lemma 1, Section 7. $\square$

Lemma 6.3.2 (Lemma 5.5 in [KP93])**.** *Let Ω be a Lipschitz domain. Suppose $u \in W^{1,2}(\Omega)$ and continuous on $\overline{T_R(Q)}$. If $u \equiv 0$ on $\Delta_R(Q)$ and $0 \leq \alpha < 1$ then*

$$\int_{T_R(Q)} \delta^\alpha(X) u^2(X) \, dX \leq C_\alpha R^2 \int_{T_R(Q)} \delta(X)^\alpha |\nabla u(X)|^2 \, dX.$$

Proof. Without losing generality, we can assume that $R \leq R_0$ and that u is smooth. By the

[2] C.E. Kenig and J. Pipher show that if f is a Lipschitz function supported in $\Delta_r(Q_0)$ (with r smaller than a fixed constant which depends on the domain Ω) and $||\nabla_T f||_{L^\infty(\partial\Omega)} \leq \frac{1}{r^{n-1}}$, then $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} \leq C$, where u is the weak solution with boundary data f.

fundamental Theorem of calculus, we have for $\gamma_X = \{Z = (1-t)X + t\hat{X} : t \in [0,1]\}$ that

$$u(X) \leq \int_{\gamma_X} |\nabla u| \left(\frac{\delta(\cdot)}{\delta(\cdot)}\right)^{\frac{\alpha}{2}}.$$

The Cauchy–Schwarz inequality and a change of variables imply

$$\begin{aligned}\int_{T_R(Q)} \delta(X)^\alpha u^2(X)\,\mathrm{d}X &\leq C_\alpha \int_{T_R(Q)} \delta(X)^\alpha \left(\int_{\gamma_X} |\nabla u|^2 \delta^\alpha\right)\left(\int_{\gamma_X} \delta^{-\alpha}(\cdot)\right)\\ &\leq C_\alpha \int_{T_R(Q)} \delta(X)^\alpha |\nabla u(X)|^2\,\mathrm{d}X.\end{aligned}$$

□

The following is proven for elliptic operators $L_0 = \mathrm{div} A\nabla$ in Lemma 5.8 and Lemma 5.13 in [KP93]. The proof works equally well for $L \in \mathcal{O}_0$. For completeness we include this proof.

Lemma 6.3.3. *Let $0 < R < \frac{1}{4}R'$ and Ω be a Lipschitz domain. Assume that u is a non-negative weak solution for $L \in \mathcal{O}_0$ which vanishes on $\Delta_{R'}(Q), Q \in \partial\Omega$. Then there exists an $\varepsilon > 0$ such that*

$$\int_{T_R(Q)} |\nabla u|^2 \leq C \int_{T_R(Q)\setminus S_{\varepsilon,R}(Q)} |\nabla u|^2.$$

For $X \in T_{\frac{1}{4}R'}(Q)$ and $\delta(X) = R$, we have

$$\frac{u(X)}{\delta(X)} \approx \left(\fint_{T_R(Q)\setminus S_{\varepsilon,R}(Q)} |\nabla u|^2\right)^{\frac{1}{2}}$$

Proof. Without losing generality, we can assume that $R \leq R_0$. According to the Hölder continuity up to the boundary, we have

$$\begin{aligned}\fint_{T_R(Q)} u(X)^2 \leq Cu(A_R(Q))^2 &\leq \frac{C}{R^\alpha} \fint_{B_{\frac{1}{10}R}(A_R(Q))} \delta(X)^\alpha u(X)^2\,\mathrm{d}X\\ &\leq \frac{C}{R^\alpha} \fint_{T_R(Q)} \delta(X)^\alpha u^2(X)\,\mathrm{d}X.\end{aligned}$$

Caccioppoli's inequality, the Hölder continuity up to the boundary, the previous estimate and Lemma 6.3.2 give us

$$\begin{aligned}\int_{T_R(Q)} |\nabla u(X)|^2 &\leq \frac{C}{R^2} \int_{T_R(Q)} u^2(X)\,\mathrm{d}X\\ &\leq \frac{C}{R^2}\frac{1}{R^\alpha} \int_{T_R(Q)} \delta(X)^\alpha u^2(X)\,\mathrm{d}X\\ &\leq \frac{C}{R^\alpha} \int_{T_R(Q)} \delta(X)^\alpha |\nabla u(X)|^2\,\mathrm{d}X\\ &= \frac{C}{R^\alpha} \int_{T_R(Q)\setminus S_{\varepsilon,R}(Q)} \delta(X)^\alpha |\nabla u(X)|^2\,\mathrm{d}X + \frac{C}{R^\alpha} \int_{S_{\varepsilon,R}(Q)} \delta(X)^\alpha |\nabla u(X)|^2\,\mathrm{d}X\\ &\leq C \int_{T_R(Q)\setminus S_{\varepsilon,R}(Q)} |\nabla u|^2 + C\varepsilon^\alpha \int_{T_R(Q)} |\nabla u|^2.\end{aligned}$$

The second term can be hidden on the left hand side, which completes the proof of the first part of the Lemma.
The Caccioppoli inequality and the Hölder continuity at the boundary imply with $R = \delta(X)$ that

$$\left(\fint_{T_R(Q)\setminus S_{\varepsilon,R}(Q)} |\nabla u|^2\right)^{\frac{1}{2}} \leq \frac{C}{R}\left(\fint_{T_{2R}(Q)} u^2\right)^{\frac{1}{2}} \leq C\frac{u(X)}{\delta(X)},$$

which is the first inequality of the second part of the Lemma. For the other direction, we use the Poincaré inequality and the Harnack principle as follows

$$\left(\fint_{T_R(Q)} |\nabla u|^2\right)^{\frac{1}{2}} \geq \frac{C}{R}\left(\fint_{T_R(Q)} u^2\right)^{\frac{1}{2}} \geq C\frac{u(X)}{\delta(X)}.$$

□

For the following theorem let us recall that $(D^*)_p = (D)_p^{L^*}$ for L^* the adjoint of L.

Theorem 6.3.4. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Then $(R)_p$ implies $(D^*)_{p'}$. Moreover $(R)_{HS^1}$ implies $w^* \in A_\infty$ and so $(D^*)_q$ for some $1 < q < \infty$.*

Proof. We will prove only the $(R)_{\mathrm{HS}^1}$ result, whereas we will mention the changes to the $(R)_p$ case where necessary. We use the methods and ideas from the proof in [KP93] and change them a bit to suit the $(R)_{\mathrm{HS}^1}$ condition. Let w^* be the elliptic measure for L^*. Then we have to show that w^* is absolutely continuous with respect to the surface measure and that $w^* \in A_\infty$ (respectively $w^* \in B_p$ for the $(R)_p$ result).
Choose $R \leq \frac{1}{5}R_0$ and $Q_0 \in \partial\Omega$. Let $f \in C^\infty(\partial\Omega)$ be non-negative with $0 \leq f \leq 1$ and

$$\begin{cases} f \equiv 0 & \text{on } \Delta_R = \Delta_R(Q_0) \\ f \equiv 1 & \text{on } \Delta_{4R}\backslash\Delta_{2R} \\ f \equiv 0 & \text{on } \partial\Omega\backslash\Delta_{5R} \end{cases}$$

such that $||\nabla f||_\infty \leq \frac{C}{R}$. We have $||f||_{L^\infty(\partial\Omega)} \leq 1$ and $||\nabla f||_{L^\infty(\partial\Omega)} \leq \frac{C}{R}$, thus $\frac{C}{R^{n-2}}f$ is a Hardy–Sobolev $(1,\infty)$-atom and therefore $||f||_{\mathrm{HS}^1} \leq CR^{n-2}$.
Let u be the weak solution with boundary data f, then as we have used before $C \leq u(A_R(Q_0)) \leq 1$. By the comparison principle and Lemma 4.2.3 we have

$$\frac{u(X)}{G(X,0)} \approx \frac{u(A_R(Q_0))}{G(A_R(Q_0),0)} \approx \frac{1}{G^*(0,A_R(Q_0))} \approx \frac{R^{n-2}}{w^*(\Delta_R)}$$

for $X \in T_{R/2}(Q_0)$. Lemma 6.3.3 and Lemma 4.2.3 imply

$$\left(\fint_{T_{\delta(X)}(\hat{X})\backslash S_{\varepsilon,\delta(X)}(\hat{X})} |\nabla u|^2\right)^{\frac{1}{2}} \approx \frac{u(X)}{\delta(X)} \approx \frac{G(X,0)}{\delta(X)}\frac{R^{n-2}}{w^*(\Delta_R)} \approx \frac{w^*(\Delta_{\delta(X)}(\hat{X}))}{\delta(X)^{n-1}}\frac{R^{n-2}}{w^*(\Delta_R)}.$$

Then, for $P = \hat{X}$, we have

$$\frac{w^*(\Delta_{\delta(X)}(P))}{\delta(X)^{n-1}} \approx \frac{w^*(\Delta_R)}{R^{n-2}}\left(\fint_{T_{\delta(X)}(\hat{X})\backslash S_{\varepsilon,\delta(X)}} |\nabla u|^2\right)^{\frac{1}{2}} \leq C\frac{w^*(\Delta_R)}{R^{n-2}}N^\varepsilon(\nabla u)(P).$$

Hence if we define $h(P) = \sup_{0<s<\frac{R}{2}} \frac{w^*(\Delta_s(P))}{s^{n-1}}$, the estimate above gives that

$$h(P) \leq C\frac{w^*(\Delta_r)}{r^{n-2}}N^\varepsilon(\nabla u)(P).$$

By Lemma 6.3.3, the assumption that $(R)_{\mathrm{HS}^1}$ holds, and the doubling property of w^*, we see that w^* is absolutely continuous with respect to $\mathrm{d}\sigma$, i.e. $w^* = k^*\,\mathrm{d}\sigma$ for some $k^* \in L^1(\mathrm{d}\sigma)$.
To establish that $w^* \in A^\infty(\mathrm{d}\sigma)$ it is enough to establish that $||k^*||_{L(logL)(\mathrm{d}\tilde{\sigma})} \leq C||k^*||_{L^1(\mathrm{d}\tilde{\sigma})}$ for all surface measures $\mathrm{d}\tilde{\sigma} = \frac{\chi_\Delta}{|\Delta|}\mathrm{d}\sigma$ with Δ a surface ball where we can assume without losing generality that $r(\Delta) \leq R_0$. Theorem 5.2.6 implies

$$||k^*||_{L(logL)(\mathrm{d}\tilde{\sigma})} \leq C||M_\Delta k^*||_{L^1(\mathrm{d}\tilde{\sigma})},$$

where M_Δ denotes the Hardy–Littlewood maximal function over all balls contained in Δ. By the doubling property of w^*, we see that

$$\begin{aligned}||M_\Delta k^*||_{L^1(\,\mathrm{d}\tilde{\sigma})} &\leq C \fint_{\Delta_{R/2}} h(P)\,\mathrm{d}\sigma(P) \\ &\leq C \frac{w^*(\Delta_R)}{R^{n-2}} \fint_{\Delta_{R/2}} N^\varepsilon(\nabla u)(P)\,\mathrm{d}\sigma \\ &\leq C \frac{w^*(\Delta_R)}{R^{n-2}} \frac{1}{R^{n-1}} R^{n-2} = C||k^*||_{L^1(\,\mathrm{d}\tilde{\sigma})},\end{aligned}$$

which concludes that $w^* \in A_\infty$. In the $(R)_p$ case, the last calculation shows that $w^* \in B_p$. □

6.3.2 A new proof for: $(R)_p$ implies $(R)_{\mathbf{HS}^1}$

In order to prove that $(R)_p$ implies $(R)_{\mathrm{HS}^1}$ without the localization Theorem of [KP93], we need the following:

Lemma 6.3.5. *Let Ω be a Lipschitz domain, $L \in \mathcal{O}_0$ and G the corresponding Green's function. Let u be a weak solution which vanishes on $\Delta_{5R}(Q)$. Then for any $X \in T_{2R}(Q)$, we have*

$$|u(X)| \approx \frac{G(X,0)}{G(A_R(Q),0)} \left(\fint_{T_{4R}(Q)} |u|^q \right)^{\frac{1}{q}},$$

for $0 < q < \infty$, where the implicit constant depends on q.

Proof. The proof is taken from the proof of Lemma 2.5 in [She07], where $q = 2$ is considered. Without losing generality, we can assume that $R \leq \frac{1}{5}R_0$. Let u_1 be the weak solution on $T_{3R}(Q)$ with boundary data $u^+ = \max\{u, 0\}$ on $\partial T_{3R}(Q)$ and u_2 be the weak solution on $T_{3R}(Q)$ with boundary data $u^- = -\min\{u, 0\}$. Then, the comparison principle for non-negative weak solutions implies for $X \in T_{2R}(Q)$ that

$$u_j(X) \leq C \frac{G(X,0)}{G(A_R(Q),0)} \sup_{\partial T_{3R}(Q)} u_j.$$

Since $u_1 = u_0^+$ and $u_2 = -u_0^-$ on $\partial T_{3R}(Q)$ (see Theorem 2.2.7 for the definition of u_0^+) we get by the maximum principle and Theorem 2.2.7:

$$\begin{aligned}|u(X)| \leq |u_1(X)| + |u_2(X)| &\leq C \frac{G(X,0)}{G(A_R(Q),0)} \sup_{\partial T_{3R}(Q)} u_0^+ + \sup_{\partial T_{3R}(Q)} (-u_0^-) \\ &\leq C \frac{G(X,0)}{G(A_R(Q),0)} \left[\left(\fint_{T_{4R}(Q)} |u_0^+|^q \right)^{\frac{1}{q}} + \left(\fint_{T_{4R}(Q)} |u_0^-|^q \right)^{\frac{1}{q}} \right] \\ &\leq C \frac{G(X,0)}{G(A_R(Q),0)} \left(\fint_{T_{4R}(Q)} |u|^q \right)^{\frac{1}{q}}.\end{aligned}$$

□

The next Lemma is partly contained in the proof of Theorem 2.9 in [She07] and will prove itself being handy:

Lemma 6.3.6. *Let Ω be a Lipschitz domain, $L \in \mathcal{O}_0$, $1 \leq p < \infty$ and $0 < q < \infty$. Assume that $w^* \in B_p$, if $1 < p < \infty$ and $w^* \in A_\infty$, if $p = 1$. Then if u is a weak solution which vanishes on $\Delta_{5R}(Q)$, we get*

$$\left(\fint_{\Delta_R(Q)} \left[\left(\frac{u}{\delta}\right)^*_R \right]^p \right)^{\frac{1}{p}} \leq \frac{C_q}{R} \left(\fint_{T_{4R}(Q)} |u|^q \right)^{\frac{1}{q}}.$$

Proof. By Lemma 6.3.5 we have

$$\left(\frac{u}{\delta}\right)^*_R(P) \leq C_q \frac{1}{G(A_R,0)} \left(\fint_{T_{4R}(Q)} |u|^q\right)^{\frac{1}{q}} \left(\frac{G(\cdot,0)}{\delta(\cdot)}\right)^*_R(P)$$

for any $P \in \Delta_R(Q)$. Lemma 4.2.3 and Theorem 3.1.4 imply

$$\frac{G(X,0)}{\delta(X)} \approx \frac{w^*(\Delta_{\delta(X)}(\hat{X}))}{\delta(X)^{n-1}}.$$

For h_R defined as h in the proof of Theorem 6.3.4, but with the truncation at height R, we get

$$\begin{aligned}\left(\fint_{\Delta_R(Q)} \left[\left(\frac{u}{\delta}\right)^*\right]^p\right)^{\frac{1}{p}} &\leq \frac{C}{G(A_R(Q),0)} \left(\fint_{T_{4R}(Q)} |u|^q\right)^{\frac{1}{q}} \left(\fint_{\Delta_R(Q)} h_R^p\right)^{\frac{1}{p}} \\ &\leq \frac{C}{G(A_R(Q),0)} \left(\fint_{T_{4R}(Q)} |u|^q\right)^{\frac{1}{q}} \frac{w^*(\Delta_R(Q))}{R^{n-1}},\end{aligned}$$

where we used the B_p condition for $1 < p < \infty$ and (5.7) if $p = 1$ for the last step. Lemma 4.2.3 then implies

$$\left(\fint_{\Delta_R(Q)} \left[\left(\frac{u}{\delta}\right)^*\right]^p\right)^{\frac{1}{p}} \leq \frac{C}{R} \left(\fint_{T_{4R}(Q)} |u|^q\right)^{\frac{1}{q}}.$$

□

The result below takes care of the estimate for the non-tangential maximal function away from the support of an $(1,\infty)$-atom.

Theorem 6.3.7. *Assume that $L \in \mathcal{O}_0$, Ω is a Lipschitz domain and $w^* \in A_\infty$. Let f be a smooth Hardy–Sobolev $(1,\infty)$-atom corresponding to the surface ball $\Delta_R(Q_0)$. Let u be the weak solution for f, then*

$$||N(\nabla u)||_{L^1(\partial\Omega\setminus\Delta_{8R}(Q_0))} \leq C$$

for a constant C independent of f and R.

Proof. Without losing generality, we can assume that $R \leq R_0$ (where R_0 depends on the domain Ω, see Lemma 2.1.1) and that f is non-negative. Since f is a smooth Hardy–Sobolev $(1,\infty)$-atom for $\Delta_R(Q)$, we have $|f| \leq \frac{C}{R^{n-2}}$. Thus for $X \in \Omega\setminus T_{2R}(Q)$, Lemma 4.2.3 and Lemma 3.1.5 imply

$$u(X) \leq CR^{2-n}w^X(\Delta_R(Q_0)) \approx G(X, A_R(Q_0)) \leq C\frac{R^\alpha}{|X-Q_0|^{n+\alpha-2}}. \tag{6.8}$$

Define $R_j = \{Q \in \partial\Omega : |Q-Q_0| \approx 2^jR\}$ for $j \geq 3$. For $Q \in R_j$ and $X \in \Gamma(Q)$ with $|X-Q| \geq 2^jR$ we have by (6.8) and Cacciopoli's inequality

$$\left(\fint_{B_{\frac{\delta(X)}{2}}(X)} |\nabla u|^2\right)^{\frac{1}{2}} \leq \frac{C}{\delta(X)} \left(\fint_{B_{\frac{\delta(X)}{2}}(X)} u^2\right)^{\frac{1}{2}} \leq \frac{C}{\delta(X)} \frac{R^\alpha}{(2^jR)^{n+\alpha-2}} \leq \frac{C}{2^{j\alpha}} \frac{1}{(2^jR)^{n-1}}.$$

Therefore,

$$\int_{R_j} N(\nabla u)(Q)\,\mathrm{d}\sigma(Q) \leq \int_{R_j} N_{2^jR}(\nabla u)(Q)\,\mathrm{d}\sigma(Q) + \frac{C}{2^{j\alpha}}.$$

The Cacciopoli inequality implies

$$\int_{R_j} N_{2^jR}(\nabla u)(Q)\,\mathrm{d}\sigma(Q) \leq C\int_{R_j} \left(\frac{u}{\delta}\right)^*(Q)\,\mathrm{d}\sigma(Q).$$

Thus, if we cover R_j with a finite number of balls Δ_α^j with radii comparable to $2^j R$ and apply Lemma 6.3.6 to each of the balls, we get

$$\int_{R_j} N_{2^j R}(\nabla u)(Q)\,\mathrm{d}\sigma(Q) \leq C(2^j R)^{n-1} \sum_\alpha \fint_{\Delta_\alpha^j} \left(\frac{u}{\delta}\right)^* \leq \frac{(2^j R)^{n-1}}{2^j R} \sum_\alpha \left(\fint_{T_{\Delta_\alpha^j}} u^2\right)^{\frac{1}{2}},$$

where $T_{\Delta_\alpha^j} = T_{r_\alpha^j}(Q_\alpha^j)$ for $r_\alpha^j = r(\Delta_\alpha^j)$ and Q_α^j the centre of Δ_α^j. Inequality (6.8) implies that each term is bounded by $\frac{R^\alpha}{(2^j R)^{n+\alpha-2}}$, thus

$$\int_{R_j} N_{2^j R}(\nabla u)(Q)\,\mathrm{d}\sigma(Q) \leq C(2^j R)^{n-2} \frac{R^\alpha}{(2^j R)^{n+\alpha-2}} \leq \frac{C}{2^{j\alpha}}.$$

Therefore, $\int_{R_j} N(\nabla u) \leq \frac{C}{2^{j\alpha}}$, which means that we can take the sum in j to get

$$\int_{\partial\Omega \setminus \Delta_{8R}(Q)} N(\nabla u) \leq C.$$

□

Thanks to Theorem 6.3.7, we now can reprove Theorem 5.2 of [KP93] for $L \in \mathcal{O}_0$ without using the localisation Theorem as in [KP93].

Theorem 6.3.8. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}_0$. Then $(R)_p$ for any $1 < p < \infty$ implies $(R)_{HS^1}$.*

Proof. By Lemma 6.2.3 it is enough to show that (6.4) holds for smooth Hardy–Sobolev $(1,\infty)$-atoms. Let f be a smooth Hardy–Sobolev $(1,\infty)$-atom corresponding to $\Delta_R(Q)$ and u the weak solution for f. Without losing generality, we can assume that $R \leq R_0$. By Theorem 6.3.4, we know that $(D^*)_p$ holds. Theorem 6.3.7 implies

$$||N(\nabla u)||_{L^1(\partial\Omega\setminus\Delta_{8R}(Q)} \leq C.$$

For the $\Delta_{8R}(Q)$ part, we use Hölder's inequality and the $(R)_p$ condition to get

$$||N(\nabla u)||_{L^1(\Delta_{8R}(Q))} \leq C|\Delta_R(Q)|^{\frac{1}{p'}}\,||N(\nabla u)||_{L^p(\Delta_{8R}(Q))} \leq C|\Delta_R(Q)|^{\frac{1}{p'}}\,||f||_{W^{1,p}(\partial\Omega)} \leq C,$$

since f is a $(1,\infty)$-atom for $\Delta_R(Q)$.
It remains to be shown that $||u||_{L^1(\Omega)} \leq C$. From (6.8), we have

$$u(X) \leq CG(X, A_R(Q))$$

for $X \in \Omega\setminus T_{2R}(Q)$. The inequalities (3.3) and (6.5) imply

$$||u||_{L^1(\Omega)} \leq C||u||_{L^1(\Omega_{R_0})} + ||N(\nabla u)||_{L^1(\partial\Omega)} \leq C,$$

which completes the proof. □

The next Lemma (with $q = 1$) is the main Lemma in [She07], namely the reverse Hölder inequality for $N(\nabla u)$. The proof from [She07] works equally well for elliptic operators $L \in \mathcal{O}_0$. We will modify the proof to allow indices smaller one.

Lemma 6.3.9. *Let $L \in \mathcal{O}_0$, Ω a Lipschitz domain and assume that $(D^*)_{p'}$ holds. Let u be a weak solution which vanishes on $\Delta_{4R}(Q)$. Then there exists an aperture $\alpha = \alpha(\Omega) > 1$ such that*

$$\left(\fint_{\Delta_R(Q_0)} N(\nabla u)^p\right)^{\frac{1}{p}} \leq C_{q,\alpha} \left(\fint_{\Delta_{4R}(Q_0)} N_\alpha(\nabla u)^q\right)^{\frac{1}{q}}$$

for $0 < q < p$.

Proof. Without losing generality, we can assume that $R \leq R_0$. Let $P \in \Delta_R(Q_0)$ and $X \in \Gamma(P)$ with $|X-P| > C_1 R$. Then there exists an aperture $\alpha = \alpha(\Omega, C_1)$ such that $X \in \Gamma_\alpha(Q)$ for all $Q \in \Delta_{4R}(Q_0)$. Therefore, $\left(\fint_{B_{\frac{\delta(X)}{2}}(X)} |\nabla u|^2\right)^{\frac{1}{2}} \leq C N_\alpha(\nabla u)(P)$ for all $P \in \Delta_{4R}(Q_0)$ and so

$$\left(\fint_{B_{\frac{\delta(X)}{2}}(X)} |\nabla u|^2\right)^{\frac{1}{2}} \leq C \left(\fint_{\Delta_{C_0 R}(Q_0)} N_\alpha(\nabla u)(P)^q\right)^{\frac{1}{q}}$$

for any $0 < q$.
One can argue that $|X-P| > C_1 R$ implies the existence of a surface ball Δ' with radius comparable to R such that $X \in \Gamma(Q)$ for all $Q \in \Delta'$ and so $(\fint_{B_{\frac{\delta(X)}{2}}(X)} |\nabla u|^2)^{\frac{1}{2}} \leq C(\fint_{\Delta_{4R}(Q_0)} N(\nabla u)^q)^{\frac{1}{q}}$, i.e. there is no need to introduce the larger aperture α in the previous step. In the following proofs we will nevertheless apply this Lemma with the larger aperture α once, in order to show how one can deal with it.
We are left with the truncation $N_{C_1 R}(\cdot)$ to consider. We use Cacciopoli's inequality and Lemma 6.3.6 to get

$$\left(\fint_{\Delta_R(Q_0)} N_{C_1 R}(\nabla u)^p\right)^{\frac{1}{p}} \leq C \left(\fint_{\Delta_R(Q_0)} \left[\left(\frac{u}{\delta}\right)^*\right]^p\right)^{\frac{1}{p}} \leq \frac{C}{R}\left(\fint_{T_{4R}(Q_0)} |u|^q\right)^{\frac{1}{q}}.$$

Due to the zero boundary condition on $\Delta_{C_0 R}(Q_0)$ we have (by (6.5)) $u(X) \leq C\delta(X) N(\nabla u)(\hat{X})$ for $X \in T_{4R}(Q_0)$. Inserting this into the previous estimate gives us

$$\left(\fint_{\Delta_R(Q_0)} N_{C_1 R}(\nabla u)^p\right)^{\frac{1}{p}} \leq \frac{C}{R}\left(\fint_{T_{4R}(Q_0)} \delta(X)^q N(\nabla u)(\hat{X})^q \, \mathrm{d}X\right)^{\frac{1}{q}} \leq C\left(\fint_{\Delta_{4R}(Q_0)} N(\nabla u)^q\right)^{\frac{1}{q}},$$

which proves the Lemma. □

6.3.3 $(R)_{\mathbf{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$

We are now ready to establish the main result of this thesis, concretely the implication that $(R)_{\mathrm{HS}^1}$ implies $(R)_p$ for some $1 < p < \infty$. In the course of thinking about this problem we discovered that there are two possible ways to establish this result. One is to adapt the proof in [KP93] where for $(R)_p$ implies $(R)_{p+\varepsilon}$ was established. The other way is motivated by the proof of the main Theorem in [She07] (adjusted with the aid of Lemma 6.1.6). We decided we prefer the second method as it avoids the use of a localization theorem and real variable techniques with rather lengthy proofs. We present this method here. We define

$$E(\lambda) = \{P \in \partial\Omega : M(N(\nabla u))(P) > \lambda\}.$$

Theorem 6.3.10. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that $(R)_{HS^1}$ holds. Choose any $p \in (1,\infty)$ for which the $(D^*)_{p'}$ condition holds. Let $f \in C^\infty(\partial\Omega)$ and u be the corresponding weak solution. Then there exist positive constants ε, η, C_0 such that*

$$|E(\tau\lambda)| \leq \varepsilon^{1+\eta}|E(\lambda)| + |\{P \in \partial\Omega : M(M(|\nabla f|)) > \gamma\lambda\}| \tag{6.9}$$

for $\lambda > \lambda_0 = C_0 \int_{\partial\Omega} N(\nabla u)$, $\gamma = \gamma(\varepsilon)$, $\tau = \varepsilon^{-\frac{1}{p}}$.

Proof. The weak $(1,1)$ inequality for the Hardy Littlewood maximal function implies

$$E(\lambda) \leq \frac{C}{\lambda}\int_{\partial\Omega} N(\nabla u) \leq \frac{C}{\lambda}\frac{\lambda_0}{C_0}.$$

Thus, by choosing $C_0 = C_0(\Omega)$ sufficiently large, we can ensure that $E(\lambda) \leq \frac{1}{2}|\Delta_{\frac{R_0}{4}}|$, where $\Delta_{\frac{R_0}{4}}$ is any surface ball with radius $R_0/4$. Thus $E(\lambda)^c \cap \Delta_{\frac{R_0}{4}} \neq \emptyset$ for $\lambda > \lambda_0$.

Let $\{Q_k\}$ be a Whitney decomposition of $E(\lambda)$, i.e.

- $E(\lambda) = \bigcup_k Q_k$,
- $\sum_k \chi_{Q_k} \leq K$,
- $3Q_k \cap E(\lambda)^c \neq \emptyset$.

To prove the lemma we will claim that it is enough to prove that

$$Q_k \cap \{M(M(|\nabla f|)) \leq \gamma\lambda\} \neq \emptyset \text{ implies } |E(\tau\lambda) \cap Q_k| \leq \varepsilon^{1+\eta}|Q_k|. \tag{6.10}$$

Because of $E(\tau\lambda) \subset E(\lambda)$ it follows that for ε small enough such that $K\varepsilon^{1+\eta} \leq \varepsilon^{1+\frac{\eta}{2}}$

$$\begin{aligned}|E(\tau\lambda)| &\leq \sum_{\{k:Q_k\cap\{M_{\frac{R_0}{2}}(M_{R_0}(|\nabla f|))\leq\gamma\lambda\}\neq\emptyset\}} |E(\tau\lambda)\cap Q_k| + |\{M(M(|\nabla f|)) \geq \gamma\lambda\}| \\ &\leq \varepsilon^{1+\frac{\eta}{2}}|E(\lambda)| + |\{M(M(|\nabla f|)) \geq \gamma\lambda\}|,\end{aligned}$$

which is the statement of the Theorem.
Hence we focus on establishing (6.10). By the properties imposed on Q_k by the Whitney decomposition we have the following for $P \in Q_k$:

$$M(N(\nabla u))(P) \leq \max\{M_{5Q_k}(N(\nabla u)), C_1\lambda\}$$

for some $C_1 = C_1(\Omega)$ depending only on the geometry of our domain. Here M_Q is a modified version of the maximal function

$$M_Q(f)(P) = \sup_{\substack{\widetilde{Q}\ni P\\ \widetilde{Q}\subset Q}} \fint_{\widetilde{Q}} |f|.$$

Take now τ larger than C_1. We see by the properties of the Whitney decomposition on Q_k that

$$|Q_k \cap E(\tau\lambda)| \leq |\{P \in Q_k : M_{5Q_k}(N(\nabla u))(P) > \tau\lambda\}|. \tag{6.11}$$

Let v be the weak solution with boundary data $\varphi(f-\alpha)$, where $\varphi \in C^\infty(\partial\Omega)$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $6Q_k$, $\operatorname{supp}\varphi \subset 10Q_k$ and $\alpha = \fint_{10Q_k} f$. Then

$$\begin{aligned}|Q_k \cap E(\tau\lambda)| &\leq |\{P \in Q_k : M_{5Q_k}[N(\nabla(u-v))] > \frac{\tau\lambda}{2}\}| \\ &\quad + |\{P \in Q_k : M_{5Q_k}[N(\nabla v)] > \frac{\tau\lambda}{2}\}| \\ &\leq \frac{C}{(\tau\lambda)^{\bar{p}}}\int_{5Q_k} N(\nabla(u-v))^{\bar{p}} + \frac{C}{\tau\lambda}\int_{5Q_k} N(\nabla v) = I + II\end{aligned}$$

by the weak $(\bar{p},\bar{p})$ and the weak $(1,1)$ inequality. We choose $\bar{p} > p$ so that $(D^*)_{\bar{p}'}$ still holds. Since $(R)_{\mathrm{HS}^1}$ holds, Lemma 6.1.6 for $q = 1$ implies for the second term

$$II \leq \frac{C}{\tau\lambda}||\varphi(f-\alpha)||_{\mathrm{HS}^1} \leq \frac{C}{\tau\lambda}|Q_k|M(M(|\nabla f|))(Q)$$

for any $Q \in 5Q_k$. Thus we can choose a Q from $Q_k \cap \{M(M(|\nabla f|) \leq \gamma\lambda\}$ to get $II \leq \frac{C\gamma}{\tau}|Q_k|$. For I, observe that $u - v - \alpha$ is a weak solution with vanishing boundary data on $6Q_k$. Lemma

6.3.9 implies

$$
\begin{aligned}
I &\leq \frac{C}{(\tau\lambda)^{\bar{p}}}|Q_k|\left(\fint_{6Q_k} N(\nabla(u-v))\right)^{\bar{p}} \\
&\leq \frac{C}{(\tau\lambda)^{\bar{p}}}|Q_k|\left[\left(\fint_{6Q_k} N(\nabla u)\right)^{\bar{p}} + \left(\fint_{6Q_k} N(\nabla v)\right)^{\bar{p}}\right] \\
&\leq \frac{C}{(\tau\lambda)^{\bar{p}}}\left[\lambda^{\bar{p}} + (\gamma\lambda)^{\bar{p}}\right]|Q_k| \leq \frac{C}{\tau^{\bar{p}}}|Q_k|.
\end{aligned}
$$

To get the last line we have used the facts that $3Q_k \cap E(\lambda)^c \neq 0$ as well as $Q_k \cap \{M(M(|\nabla f|)) \leq \gamma\lambda\} \neq \emptyset$. In the last step we hid γ into a generic constant C, we can do this since $\gamma > 0$ will be chosen small in the next step. Collecting all estimates together, we can see that (where $q = 1$, but it will be smaller in a later proof):

$$
\begin{aligned}
|Q_k \cap E(\tau\lambda)| &\leq |Q_k|\left(\frac{C\gamma}{\tau} + \frac{C}{\tau^{\frac{\bar{p}}{q}}}\right) \\
&= |Q_k|(C\gamma\varepsilon^{\frac{q}{p}} + C\varepsilon^{\frac{\bar{p}}{qp}}) \\
&= |Q_k|\varepsilon^{1+\eta}(C\gamma\varepsilon^{\frac{q}{p}-1-\eta} + C\varepsilon^{\eta}),
\end{aligned}
$$

for $\eta = \frac{1}{2}(\frac{\bar{p}}{qp} - 1) > 0$. Now choose ε small enough to make the second term smaller than $\frac{1}{2}$ and then choose γ such that the first term is smaller than $\frac{1}{2}$. Therefore,

$$
|Q_k \cap E(\tau\lambda)| \leq \varepsilon^{1+\eta}|Q_k|,
$$

which finishes the proof. □

With (6.9), the proof of the Main Theorem in [She07] implies our main result. For completeness we include the proof.

Theorem 6.3.11. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Then there exists $1 < p < \infty$ such that $(R)_{HS^1}$ implies $(R)_p$.*

Proof. By Theorem 6.3.4, there exists $1 < p < \infty$ such that $(D^*)_{p'}$ holds. We multiply (6.9) on both sides with λ^{p-1} and then integrate over (λ_0, Λ) to get

$$
\int_{\lambda_0}^{\Lambda} |E(\tau\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda \leq \varepsilon^{1+\eta}\int_{\lambda_0}^{\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda + C\int |\nabla f|^p.
$$

For the last term we used the boundedness of the Hardy Littlewood maximal function on L^p twice. Using the change of variables $\tau\lambda \mapsto \lambda$, we get

$$
\int_{\tau\lambda_0}^{\tau\Lambda} |E(\lambda)|\lambda^{p-1}\tau^{1-p}\tau^{-1}\,\mathrm{d}\lambda \leq \varepsilon^{1+\eta}\int_{\lambda_0}^{\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda + C\int |\nabla f|^p.
$$

By the definition of τ we have $\tau^{1-p}\tau^{-1} = \varepsilon$. Therefore, the previous inequality simplifies to

$$
\int_{\tau\lambda_0}^{\tau\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda \leq \varepsilon^{\eta}\int_{\lambda_0}^{\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda + C\int |\nabla f|^p.
$$

For ε small enough such that $\varepsilon^\eta \leq \frac{1}{2}$ and Λ large enough such that $\Lambda \geq \tau\lambda_0$, we can hide the part $\varepsilon^\eta \int_{\tau\lambda_0}^{\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda$ on the left hand side to get

$$
\int_{\tau\lambda_0}^{\tau\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda \leq C\int_{\lambda_0}^{\tau\lambda_0} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda + C\int |\nabla f|^p.
$$

By adding $\int_0^{\tau\lambda_0} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda$ on both sides, we are left with

$$\int_0^{\tau\Lambda} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda \leq C\int_0^{\tau\lambda_0} |E(\lambda)|\lambda^{p-1}\,\mathrm{d}\lambda + C\int |\nabla f|^p. \tag{6.12}$$

By the definition of λ_0, the $(R)_{\mathrm{HS}^1}$-condition and Hölder's inequality, the first term of the right hand side is bounded by

$$C\left(\int_{\partial\Omega} N(\nabla u)\right)^p \leq C||f||_{\mathrm{HS}^1}^p \leq C||f||_{W^{1,p}(\partial\Omega)}^p,$$

where for the last inequality we used the fact that $||f||_{\mathrm{HS}^1} \approx ||f||_{\mathcal{C}^1}$ and Hölder's inequality. Thus sending $\Lambda \to \infty$ in (6.12) gives $\int_{\partial\Omega}(M(N(\nabla u)))^p \leq C||f||_{W^{1,p}(\partial\Omega)}^p$, i.e.

$$||N(\nabla u)||_{L^p(\partial\Omega)} \leq C||f||_{W^{1,p}(\partial\Omega)}. \tag{6.13}$$

It remains to be verified that $||u||_{L^p(\Omega)} \leq C||f||_{W^{1,p}(\partial\Omega)}$. By the usual split into the positive end negative part, we can assume that f is non-negative without losing generality. By (6.5) and (6.13), it suffices to show that $||u||_{L^p(\Omega_{\frac{R_0}{2}})} \leq C||f||_{W^{1,p}(\partial\Omega)}$. The interior Harnack principle and the $(R)_{\mathrm{HS}^1}$ condition imply

$$||u||_{L^p(\Omega_{\frac{R_0}{2}})} \leq C||u||_{L^1(\Omega_{\frac{R_0}{2}})} \leq C||f||_{\mathrm{HS}^1} \leq C||f||_{W^{1,p}(\partial\Omega)}.$$

Thus, the Theorem is proven. □

The p in Theorem 6.3.11 was determined by the p' for which $(D^*)_{p'}$ holds. Thus Theorem 6.3.11 allows us to conclude the following:

Corollary 6.3.12. *Let Ω be a Lipschitz domain and $L \in \mathcal{O}_0$ be an elliptic operator with the elliptic measure of the adjoint L^* operator in A_∞. Then either*

$$\begin{cases} (i)_a \ (D^*)_{p'} \text{ implies } (R)_p \text{ for all } p \in (1,\infty) \text{ for which } (D^*)_{p'} \text{ holds} \\ (i)_b \ w^* \in A_\infty \text{ implies } (R)_{HS^1} \end{cases}$$

or

$$\begin{cases} (ii)_a \ (R)_p \text{ is not solvable for any } p \in (1,\infty) \\ (ii)_b \ (R)_{HS^1} \text{ is not solvable.} \end{cases}$$

The question remains whether the second alternative in Corollary 6.3.12 does happen or whether $(D^*)_{p'}$ always implies $(R)_p$. By Corollary 6.3.12, Theorem 6.3.7, the part of the proof of Theorem 6.3.8 regarding the $||u||_{L^1(\Omega)}$ norm and Lemma 6.2.3, we get the following:

Corollary 6.3.13. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that for all smooth Hardy–Sobolev $(1,\infty)$-atoms f and corresponding weak solutions u, one has*

$$\int_{8\Delta_R(Q)} N(\nabla u) \leq C,$$

where $\Delta_R(Q)$ is a surface ball on which the atom f is supported and C is a constant independent of f. Then

$$(D^*)_{p'} \quad \text{implies} \quad (R)_p.$$

6.3.4 The $(R)_{\mathcal{C}^q}$ condition for $q < 1$

In this part, we will consider the problem of extending the $(R)_p$ condition to $p < 1$ in the way that $(R)_q$ and $(D^*)_{p'}$ for $1 < p < \infty$ imply $(R)_p$. For this, we define

Definition 6.3.1. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. We say that $(R)_{\mathcal{C}^q}$ for $0 < q < 1$ holds if the weak solution u to the problem*

$$\begin{aligned} Lu &= 0 \ \text{in } \Omega \\ u &\equiv f \ \text{on } \partial\Omega \end{aligned}$$

with $f \in \mathcal{C}^q \cap C^0(\partial\Omega)$ satisfies

$$||N(\nabla u)||_{L^q(\partial\Omega)} + ||u||_{L^q(\Omega)} \leq C||f||_{\mathcal{C}^q}$$

for a C independent of f.

Theorem 6.3.14. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. For $0 < q < 1$ and $1 < p < \infty$, we get that*

$$(R)_{\mathcal{C}^q} \text{ and } (D^*)_{p'} \text{ imply } (R)_p.$$

i.e. $(D^*)_{p'}$ implies $(R)_p$ or $(R)_q$ does not hold for any $0 < q < \infty$ with $(R)_1 = (R)_{HS^1}$ and $(R)_q = (R)_{\mathcal{C}^q}$ for $0 < q < 1$.

The proof follows the same lines as the proof of Theorem 6.3.11, whereas we use the result for $q < 1$ in Lemma 6.3.9 and Lemma 6.1.6. We define similarly to before

$$\begin{aligned} E(\lambda) &= \{Q \in \partial\Omega : M(N(\nabla u)^q)(Q) > \lambda\} \\ E_\alpha(\lambda) &= \{Q \in \partial\Omega : M(N_\alpha(\nabla u)^q)(Q) > \lambda\}. \end{aligned}$$

Lemma 6.3.15. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that $(R)_{\mathcal{C}^q}$ and $(D^*)_{p'}$ hold for some $0 < q < 1 < p < \infty$. Then there exist $C_0, \eta, \varepsilon > 0, \alpha$ such that*

$$|E(\tau\lambda)| \leq \varepsilon^{1+\eta}|E_\alpha(\lambda)| + |\{M[M(\nabla f)^q] > \gamma\lambda\}|$$

for $\lambda > \lambda_0 = C_0 \int_{\partial\Omega} N_\alpha(\nabla u)^q \, d\sigma, \tau = \varepsilon^{-\frac{q}{p}}, \gamma = \gamma(\varepsilon)$.

Proof. We will only point out the differences to the proof of (6.9). The Whitney decomposition is applied to $E_\alpha(\lambda)$. According to the definition of N and N_α we have $N(\nabla u) \leq N_\alpha(\nabla u)$. Since Q_k is maximal with respect to $N_\alpha(\nabla u)$ we have

$$M(N(\nabla u)^q)(P) \leq \max\{M_{5Q_k}(N(\nabla u)^q), C_1\lambda\}$$

for $P \in Q_k$. Thus, as in (6.11), we get the following for τ large enough:

$$|Q_k \cap E(\tau\lambda)| \leq |\{P \in Q_k : M_{5Q_k}(N(\nabla u))^q > \tau\lambda\}|.$$

By introducing the same v as before, one is left with

$$|Q_k \cap E(\tau\lambda)| \leq \frac{C}{(\tau\lambda)^{\frac{\bar{p}}{q}}} \int_{5Q_k} N(\nabla(u-v))^{\bar{p}} + \frac{C}{\tau\lambda} \int_{5Q_k} N(\nabla v)^q = I + II,$$

where the weak $(\frac{\bar{p}}{q}, \frac{\bar{p}}{q})$ and weak $(1,1)$ inequality are used. II is treated as before, where one uses the $(R)_{\mathcal{C}^q}$ condition instead of the $(R)_{\mathrm{HS}^1}$ condition. For I one uses Lemma 6.3.9 to get

$$I \leq \frac{C|Q_k|}{(\tau\lambda)^{\frac{\bar{p}}{q}}} \left[\left(\fint_{C_0Q_k} N_\alpha(\nabla u)^q(P) \, \mathrm{d}P \right)^{\frac{\bar{p}}{q}} + \left(\fint_{C_0Q_k} N_\alpha(\nabla v)^q(P) \, \mathrm{d}P \right)^{\frac{\bar{p}}{q}} \right].$$

The first integral is bounded by $\lambda^{\frac{\bar{p}}{q}}$ by the Whitney decomposition properties. For the second integral, one observes that $||N_\alpha(\cdot)||_{L^q(\partial\Omega)} \leq C_{\alpha,q}||N(\cdot)||_{L^q(\partial\Omega)}$ for all $0 < q < \infty$ (see for example equation (25) on page 62 in [Ste93]) and so

$$\int_{\partial\Omega} N_\alpha(\nabla v)^q \leq C_{\alpha,q} \int N(\nabla v)^q \leq C|Q_k|(\gamma\lambda)^{\frac{\bar{p}}{q}}$$

by (6.1.6). From here, one can finish the proof as for (6.9). $\square$

If we compare Lemma 6.3.15 with (6.9), we can see that we have $E_\alpha(\lambda)$ instead of $E(\lambda)$. The following Lemma will remove this issue:

Lemma 6.3.16. *Using the same setting as for Lemma 6.3.15, we have* $|E_\alpha(\lambda) \leq C_\alpha|E(\lambda)|$.

Proof. We know that

$$|\{P \in \partial\Omega : N_\alpha(\nabla u)^q(P) > \lambda\}| \leq C_\alpha |\{P \in \partial\Omega : N(\nabla u)^q(P) > \lambda\}| \tag{6.14}$$

(see for example equation (25), page 62 in [Ste93]). Applying (5.4) and (6.14) to $E_\alpha(\lambda)$, we get

$$\begin{aligned}
|E_\alpha(\lambda)| &\leq \frac{C}{\lambda} \int_{\{N_\alpha(\nabla u)^q > \frac{\lambda}{2}\}} N_\alpha(\nabla u)^q \\
&= \frac{C}{\lambda}\left[\int_{\frac{\lambda}{2}}^{\infty} |\{N_\alpha(\nabla u)^q > t\}| \,\mathrm{d}t + \frac{\lambda}{2}|\{N_\alpha(\nabla u)^q > \frac{\lambda}{2}\}|\right] \\
&\leq \frac{C_\alpha}{\lambda}\left[\int_{\frac{\lambda}{2}}^{\infty} |\{N(\nabla u)^q > t\}| \,\mathrm{d}t + \frac{\lambda}{2}|\{N(\nabla u)^q > \frac{\lambda}{2}\}|\right] \\
&\leq \frac{C_\alpha}{\lambda}\left[\int_{\{N(\nabla u)^q > \frac{\lambda}{2}\}} N(\nabla u)^q\right] \\
\overset{(5.5)}{} &\leq C_\alpha |\{M[N(\nabla u)^q] > \lambda\}| = C_\alpha |E(\lambda)|.
\end{aligned}$$

$\square$

Proof of Theorem 6.3.14. For ε small enough, Lemma 6.3.16 implies that

$$|E(\tau\lambda)| \leq \varepsilon^{1+\eta'}|E(\lambda)| + |\{M[M(|\nabla f|)]^q > \gamma\lambda\}|.$$

Thus one can repeat the proof of Theorem 6.3.11. $\square$

6.3.5 A uniform bound for $(u)_r$ in HS[1]

Going back to Theorem 6.2.1 and Theorem 6.2.2, we see that if $1 < p < \infty$, then $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} < \infty$ implies the existence of non-tangential limits $u|_{\partial\Omega}$ with $u|_{\partial\Omega}$ in $W^{1,p}(\partial\Omega)$. The proof relies on the boundedness of the maximal function in L^p. In addition, one can show by the boundedness of the maximal function that $(u)_r \in W^{1,p}(\partial\Omega)$ uniformly in r.[3] We have

$$\begin{aligned}
&\int_{\partial\Omega}\left(\sup_s \frac{1}{s} \fint_{\Delta_s(Q)} |(u)_r(P) - \operatorname{avg}_{\Delta_s(Q)}(u)_r| \,\mathrm{d}P\right)^p \mathrm{d}Q \\
&\leq C\int_{\partial\Omega}\left(\sup_s \fint_{\Delta_s(Q)} |\nabla_T(u)_r(P)| \,\mathrm{d}P\right)^p \mathrm{d}Q \\
&\leq C\int_{\partial\Omega}\left(\sup_s \fint_{\Delta_s(Q)} (|\nabla u|)_{r'}(P) \,\mathrm{d}P\right)^p \mathrm{d}Q \\
&\leq C\int_{\partial\Omega} (|\nabla u|)_{r'}(Q)|^p \,\mathrm{d}Q \leq C\int_{\partial\Omega} N(\nabla u)^p,
\end{aligned}$$

where the second inequality follows from

Lemma 6.3.17. *Let* $u \in W^{1,p}(\mathbb{R}^n_+), 1 \leq p < \infty$. *Then, for* $0 < r < r' < (1 + \frac{1}{10})r$ *and* $P \in \partial\mathbb{R}^n_+$ *we have*

$$|\nabla_T(u)_r(P)| \leq C(|\nabla u|)_{r'}(P).$$

[3] $(u)_r$ was defined by $(u)_r(Q) = \fint_{B_{\frac{r}{2}}(A_r(Q))} u(X)\,\mathrm{d}X$. In addition we define $(u)_{r'} = \fint_{B_{\frac{3r}{4}}(A_r(Q))} u(X)\,\mathrm{d}X$.

Proof. We have

$$
\begin{aligned}
|\nabla_T (u)_r(P)| &\le \sup_{0<h<\frac{1}{10}r} \Big|\frac{1}{h}\int_{B_{\frac{1-r}{2}}(rP+h\vec{T})} u(X)\,\mathrm{d}X - \int_{B_{\frac{1-r}{2}}(rP)} u(X)\,\mathrm{d}X\Big| \\
&= \sup_{0<h<\frac{1}{10}r} \Big|\int_{B_{\frac{1-r}{2}}(rP)} \frac{u(X+h\vec{T})-u(X)}{h}\,\mathrm{d}X\Big| \\
&\le \int_{B_{\frac{1-r}{2}+\frac{1}{10}r}(rP)} |\nabla u(X)|\,\mathrm{d}X,
\end{aligned}
$$

where we used Theorem 3 in [Eva98], Chapter 5.8, for the last step. □

Thus, if $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} < \infty$, the uniform bound on $(u)_r$ in $W^{1,p}(\partial\Omega)$ leads to the weak convergence of $(\nabla_T u)_r$ to $\nabla_T u|_{\partial\Omega}$, which is a stronger result than the convergence result proven in Theorem 6.2.2. Thus, we will look at the question if $(u)_r \in \mathrm{HS}^1$ uniformly in r.

Lemma 6.3.18. *Let $L \in \mathcal{O}_0$ and Ω be a Lipschitz domain. Assume that $(R)_{HS^1}$ holds. Let $f \in HS^1$ and u be the corresponding weak solution. Then*

$$||(u)_r||_{HS^1} \le C||f||_{HS^1},$$

C being a constant independent of r and f.

Proof. By (6.5) and $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} \le C||f||_{\mathrm{HS}^1}$, we have $||(u)_r||_{L^1(\partial\Omega)} \le C||f||_{\mathrm{HS}^1}$. Thus it remains to be shown that $||((u)_r)_1^b||_{L^1(\partial\Omega)} \le C||f||_{\mathrm{HS}^1}$, where $(\cdot)_1^b$ is the maximal function defined in Definition 6.1.1.
Without losing generality, we can assume that f is a Hardy–Sobolev $(1,\infty)$-atom corresponding to $\Delta_R(Q_0)$ with $R \le R_0$. Then we have to show that

$$\int_{\partial\Omega} \sup_{0<s} \left(\frac{1}{s}\int_{\Delta_s(Q)} |(u)_r(P) - \mathrm{avg}_{\Delta_s(Q)}(u)_r|\,\mathrm{d}P\right)\mathrm{d}Q \le C$$

for some C independent of f. We split the integral into two parts, namely

$$\int_{\partial\Omega} \sup_{0<s}(\dots) = \int_{\Delta_{10R}(Q_0)} \sup_{0<s}(\dots) + \int_{\partial\Omega\setminus\Delta_{10R}(Q_0)} \sup_{0<s}(\dots) = A + B.$$

We start with dealing with A:

$$
\begin{aligned}
A &\le C\left[\int_{\Delta_{10R}(Q_0)} \left(\sup_{0<s} \fint_{\Delta_s(Q)} (|\nabla u|)_{r'}\right)^p\right]^{\frac{1}{p}} |\Delta_R(Q_0)|^{\frac{1}{p'}} \\
&\le C\left[\int_{\Delta_{10R}(Q_0)} \left(\sup_{0<s} \fint_{\Delta_s(Q)} N(\nabla u)\right)^p\right]^{\frac{1}{p}} |\Delta_R|^{\frac{1}{p'}} \\
&\le C_p \left(\int_{\partial\Omega} N(\nabla u)^p\right)^{\frac{1}{p}} |\Delta_R|^{\frac{1}{p'}} \le C||f||_{W^{1,p}(\partial\Omega)}|\Delta_R|^{\frac{1}{p'}},
\end{aligned}
$$

where, for the last inequality, we used the fact that $(R)_p$ holds by Theorem 6.3.11. Since $||f||_{W^{1,p}(\partial\Omega)} \le C|\Delta_R|^{\frac{1}{p}}|\Delta_R|^{-1}$, we have $A \le C$. Thus we are left with B. We split B into

$$B \le \int_{\partial\Omega\setminus\Delta_{10R}(Q_0)} \sup_{0<s<r}(\dots) + \int_{\partial\Omega\setminus\Delta_{10R}(Q_0)} \sup_{r<s}(\dots) = I + II.$$

For I, we use the geometric properties of $N(\cdot)$. For $s < r$, we have

$$\frac{1}{s}\fint_{\Delta_s(Q)} |(u)_r - \operatorname{avg}_{\Delta_s(Q)}(u)_r| \leq C \fint_{\Delta_s(Q)} (|\nabla u|)_{r'} \leq C N_\alpha(\nabla u)(Q)$$

for an appropriately large aperture α. Thus

$$I \leq C \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} N_\alpha(\nabla u)(Q)\,\mathrm{d}Q \leq C_\alpha ||f||_{\mathrm{HS}^1} \leq C.$$

We decompose II even further:

$$\begin{aligned} II &\leq \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \sup_{r<s<R}(\dots) + \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \sup_{r<R<s}(\dots) + \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \sup_{R<r<s}(\dots) \\ &= II_A + II_B + II_C. \end{aligned}$$

For II_A, we would like to use the reverse Hölder condition of Lemma 6.3.9. Let Δ_R^j be a cover of $\partial\Omega\backslash\Delta_{10R}(Q_0)$ of surface balls with radius R and finite overlap. Then

$$\begin{aligned} II_A &\leq C \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \left(\sup_{r<s<R} \fint_{\Delta_s(Q)} (|\nabla u|)_{r'}(P)\,\mathrm{d}P \right) \mathrm{d}Q \\ &\leq C \sum_j |\Delta_R^j| \left(\fint_{\Delta_R^j} \left[\sup_{r<s<R} \fint_{\Delta_s(Q)} N_{r'}(\nabla u) \right]^p \right)^{\frac{1}{p}} \\ &\leq C_p \sum_j |\Delta_R^j| \left(\fint_{2\Delta_R^j} N_{r'}(\nabla u)^p \right)^{\frac{1}{p}} \\ &\leq C \sum_j |\Delta_R^j| \fint_{2\Delta_R^j} N_{r'}(\nabla u) \leq C ||f||_{\mathrm{HS}^1} \leq C. \end{aligned}$$

The terms II_B and II_C are both treated in a similar manner. We will only give a proof for II_B and mention the differences for II_C. We split II_B into

$$II_B = \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \sup_{r<R<s} \frac{1}{s}\frac{1}{|\Delta_s|} \left(\int_{\Delta_s(Q)\backslash\Delta_{5R}(Q_0)} (u)_r + \int_{\Delta_{5R}(Q_0)\cap\Delta_s(Q)} (u)_r \right) = II_{B_1} + II_{B_2}.$$

Since f is a Hardy–Sobolev $(1,\infty)$-atom for $\Delta_R(Q)$, $u \leq \frac{C}{R^{n-2}}$ according to the maximum principle. For $\Delta_{5R}(Q_0) \cap \Delta_s(Q) \neq \emptyset$ with $s > R$, one needs $|Q_0 - Q| \leq Cs$. Thus by writing $\partial\Omega\backslash\Delta_{10R}(Q_0) = \bigcup_j R_j$ with $R_j = \{Q \in \partial\Omega : |Q - Q_0| \approx 2^j R\}$, we get

$$\begin{aligned} II_{B_2} &\leq \sum_j |R_j| \frac{1}{2^j R} \frac{1}{(2^j R)^{n-1}} ||u||_{L^\infty(\Omega)} |\Delta_R(Q_0)| \\ &\leq \sum_j (2^j R)^{n-1} \frac{1}{(2^j R)^n} \frac{1}{R^{n-2}} R^{n-1} \leq C. \end{aligned}$$

For II_{B_1} we use the fact that $u \equiv 0$ on $\partial\Omega\backslash\Delta_{10R}(Q_0)$ and (6.5) to get

$$\begin{aligned} II_{B_1} &\le C \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \left(\sum_{j\ge 0} \frac{1}{2^j R} \fint_{\Delta_{2^jR}(Q)\backslash\Delta_{10R}(Q_0)} (u)_r(P)\,\mathrm{d}P \right) \mathrm{d}Q \\ &\le C \int_{\partial\Omega\backslash\Delta_{10R}(Q_0)} \left(\sum_{j\ge 0} \frac{1}{2^j R} r \fint_{\Delta_{2^jR}(Q)} N(\nabla u)(P)\,\mathrm{d}P \right) \mathrm{d}Q \\ &\le C \sum_{j\ge 0} \frac{r}{2^j R} \int_{\partial\Omega} N(\nabla u) \le C\frac{r}{R} ||f||_{\mathrm{HS}^1} \le C\frac{r}{R}. \end{aligned}$$

Since $r < R$ in the II_{B_1} case, this shows that $II_{B_1} \le C$. In the II_{C_1} case, the sum over j in the last calculation starts at k_0 with $2^{k_0}R \approx r$. At the end, one is left with $C\frac{r}{2^{k_0}R} \approx C$. Hence the term II_{C_2} is also bounded by a constant, which completes the proof. □

6.3.6 A note on the Neumann Problem

So far, we have been interested in the Dirichlet and regularity problem. Both of them are of the same nature in the sense that the function one is looking for should have some certain values at the boundary. Moreover, the regularity problem can be seen as the question if one can control the non-tangential derivatives of a solution by the tangential derivatives. If this question is reversed, we are asking if one can control the tangential derivatives by the non-tangential derivatives, which leads to the Neumann problem.
To focus on the idea, the domain will this time be the unit ball B and the class of elliptic operators will be all elliptic operators L_0 of the form $L_0 = \mathrm{div}A\nabla$ for A as in the definition of $\mathcal{O}$ and symmetric. Let $\vec{N}(Q)$ be the unit inward normal at $Q \in \partial B$. The Neumann problem is the problem to find a solution u such that

$$\begin{aligned} L_0 u &= 0 \text{ in } B \\ A\nabla u \cdot \vec{N}(Q) &= f \text{ on } \partial B. \end{aligned}$$

If $f \in L^2(\partial B)$, the Lax-Milgram Theorem implies that the Neumann problem has a variational solution, i.e. that one can find a unique function $u \in W^{1,2}(B)$ such that

$$\int_B A\nabla u \cdot \nabla\varphi = \int_{\partial B} f\varphi\,\mathrm{d}\sigma$$

for all $\varphi \in \mathrm{Lip}(\bar{B})$, $\mathrm{Lip}(\bar{B})$ being the space of Lipschitz functions on $\bar{B}$.
In [KP93], C.E. Kenig and J. Pipher are interested in unique solutions (modulo constants) for boundary data $f \in L^p(\partial\Omega)$ and in a non-tangential estimate to guarantee convergence of the non-tangential derivatives of the weak solution to the boundary data. C.E. Kenig and J. Pipher define:

Definition 6.3.2 (Definition (N) in [KP93])**.** *Let μ be a finite measure on ∂B. We say that $U \in L^1(B)$ is a weak solution to the Neumann problem with data μ if*

$$\int_B U\,L\varphi = \int_{\partial B} \varphi\,d\mu - \int_{\partial B} d\mu \fint_{\partial B} \varphi\,d\sigma$$

for all $\varphi \in C^0(\bar{B}) \cap W^{1,2}(B)$ such that $L\varphi \in C^0(\bar{B})$ and such that φ satisfies

$$\int_B \theta\,L\varphi + \int_B A\nabla\theta \cdot \nabla\varphi = 0$$

for all $\theta \in W^{1,2}(B)$.

C.E. Kenig and J. Pipher point out that if $\mu \in (W^{\frac{1}{2},2}(\partial B))^*$, a variational solution with data $\mu - \fint_{\partial B} \mathrm{d}\mu$ is a weak solution to the Neumann problem with data μ. They define the $(N)_p$

condition, which is the equivalent of the $(R)_p$ and $(D)_p$ condition, for the Neumann problem as follows:

Definition 6.3.3 (Definition 3.6 in [KP93]). *The Neumann problem for L_0 with data in $L^p(\partial B)$, $1 < p < \infty$, abbreviated $(N)_p$, is solvable, if, whenever for $f \in L^2(\partial B) \cap L^p(\partial B)$ and $\int_{\partial B} f = 0$ the weak solution u of*

$$\int_B A\nabla u \cdot \nabla\varphi = \int_{\partial B} \varphi f$$

for all $\varphi \in W^{1,2}(B)$ (which is also the variational solution since $\int f = 0$ and $f \in L^2(\partial B)$), satisfies the estimate $||N(\nabla u)||_{L^p(\partial B)} \leq C||f||_{L^p(\partial B)}$.

For the endpoint definition of the $(N)_p$ condition, we define

Definition 6.3.4. *The Neumann problem with data in $H^1(\partial B)$, abbreviated $(N)_{H^1}$, is solvable if, whenever $f \in L^2(\partial B) \cap H^1(\partial B)$, the weak solution u of*

$$\int_B A\nabla u \cdot \nabla\varphi = \int_{\partial B} \varphi f \, d\sigma$$

for all $\varphi \in W^{1,2}(B)$ satisfies the estimate $||N(\nabla u)||_{L^1(\partial\Omega)} \leq C||f||_{H^1(\partial B)}$.

The following is the equivalent of Theorem 6.2.1 for the Neumann problem:

Theorem 6.3.19 (Theorem 3.7 in [KP93]). *Suppose $(N)_p$, $1 < p < \infty$, is solvable for L_0. Then given $f \in L^p(\partial B)$ with $\int_{\partial B} f = 0$, the function $u(X) = \int_{\partial B} N(X,Q)f(Q)\, d\sigma$, where $N(\cdot,\cdot)$ is the Neumann function as it is defined in Definition 2.5 in [KP93], satisfies the following:*

- *$L_0 u = 0$ in B*
- *$||u||_{L^p(B)} + ||N(\nabla u)||_{L^p(\partial B)} \leq C||f||_{L^p(\partial B)}$*
- *$\int_B A\nabla u \cdot \nabla\varphi = \int_{\partial B} \varphi f$ for all $\varphi \in Lip(\bar{B})$*
- *u satisfies Definition (N)*
- *$(A\nabla u \cdot \vec{N})_r \to f$ weakly in $L^p(\partial B)$.*

In [KP93] Theorem 6.2, it was proven by a localization result for the Neumann problem that

Theorem 6.3.20 (Theorem 6.2 in [KP93]). *Assume that $(N)_p$ and $(R)_p$ hold for $1 < p < \infty$. Then $(N)_q$ holds for all $1 \leq q \leq p$ where $(N)_1$ is to be understood as $(N)_{H^1}$.*

The new proof we gave in Theorem 6.3.8 for $(R)_p$ implying $(R)_{\mathrm{HS}^1}$, which is not based on the localization result for the regularity problem, does not work in the Neumann case. One reason why the Neumann localization result is so useful is the fact that zero Neumann boundary values on a part of the boundary do not unlock the useful results given by zero Dirichlet boundary values, compare section 2.3.
Furthermore, in [KP93], the extrapolation property was proven by the localization result for the Neumann problem.

Theorem 6.3.21 (Theorem 6.3 in [KP93]). *Suppose that $(N)_p$ and $(R)_p$ are solvable for some $1 < p < \infty$. Then, there exists $\varepsilon > 0$ such that $(N)_q$ holds for all $q \in [p, p+\varepsilon)$.*

Thus, as in the Regularity problem case, it is an interesting question to ask whether $(R)_{\mathrm{HS}^1}$ and $(N)_{H^1}$ imply $(N)_{1+\varepsilon}$ for some small $\varepsilon > 0$.
We will not be able to answer this question in full generality, but we can answer it with yes in two dimensions. The proof is based on the idea of a conjugate (e.g. see [KR09], page 133, (2.9)): let u be a weak solution to $\mathrm{div} A\nabla u = 0$. Then $A\nabla u$ is divergence free and can therefore be written as the curl of a function $\tilde{u}$, i.e. there exists $\tilde{u}$ such that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\tilde{u} = A\nabla u. \tag{6.15}$$

The function $\tilde{u}$ is defined by (6.15) up to a constant. In addition, $\operatorname{div} \frac{A^T}{\det A} \nabla \tilde{u} = 0$. To see this, observe that matrix multiplication gives us

$$\frac{1}{\det A} A^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and therefore,

$$\operatorname{div} \frac{A^T}{\det A} \nabla \tilde{u} = \operatorname{div} \frac{A^T}{\det A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A \nabla u = \operatorname{div} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla u = 0,$$

since the divergence of the curl of a function vanishes. For $Q \in \partial B$, let $\vec{N}(Q)$ be the normal at Q and $\vec{T}(Q)$ be the tangent. Then $\vec{T}(Q) = \vec{N}(Q) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and therefore,

$$\vec{N}(Q) \cdot A \nabla u = \vec{T}(Q) \cdot \nabla \tilde{u}.$$
$$\vec{N}(Q) \cdot \frac{A^T}{\det A} \nabla \tilde{u} = -\vec{T}(Q) \cdot \nabla u,$$

i.e. the co-normal derivative of u is the tangential derivative of $\tilde{u}$ and vice versa (modulo signs). Let K be a measurable set in B with $|K| > 0$, then $(\fint_K |\nabla \tilde{u}|^2)^{\frac{1}{2}} \approx (\fint_K |\nabla u|^2)^{\frac{1}{2}}$. To see this, write $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$\fint_K |\nabla \tilde{u}|^2 = \fint_K (M \nabla \tilde{u}) \cdot (M \nabla \tilde{u}) = \fint_K A \nabla u \cdot M \nabla \tilde{u} \le C \left(\fint_K |\nabla \tilde{u}|^2 \right)^{\frac{1}{2}} \left(\fint_K |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Thus $(\fint_K |\nabla \tilde{u}|^2)^{\frac{1}{2}} \le C (\fint_K |\nabla u|^2)^{\frac{1}{2}}$ where the constant depends only on the L^∞ norm of A. For the other direction, we use ellipticity:

$$\fint_K |\nabla u|^2 \le C_\lambda \fint_K A \nabla u \cdot \nabla u = C_\lambda \fint_K M \nabla \tilde{u} \cdot \nabla u \le C_\lambda \left(\fint_K |\nabla \tilde{u}|^2 \right)^{\frac{1}{2}} \left(\fint_K |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Thus $(\fint_K |\nabla u|^2)^{\frac{1}{2}} \le C (\fint_K |\nabla \tilde{u}|^2)^{\frac{1}{2}}$. This implies that $N(\nabla u) \approx N(\nabla \tilde{u})$.
The following lemma is the equivalent of Lemma 6.1.6 for Hardy spaces:

Lemma 6.3.22. *Fix $\frac{n}{n+1} < q \le 1$ and $R > 0$. Assume that $g \in H^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function for $B_R(0)$, i.e. $0 \le \varphi \le 1$, $\operatorname{supp} \varphi \subset B_{2R}(0)$ and $\varphi \equiv 1$ on $B_R(0)$. Define $\alpha = \frac{1}{\int \varphi} \int \varphi g$, i.e. α is the constant such that $\int \varphi (g - \alpha) = 0$. Then, we have*

$$\frac{1}{R^n} ||\varphi(g - \alpha)||_{H^q}^q \le C M([Mg]^q)(z)$$

for any $z \in B_{10R}(0)$.

Proof. Let $\phi \in \mathcal{S}, \phi \ge 0$ with $\int \phi = 1$ (i.e. ϕ is a test function used for the maximal characterization of the Hardy Spaces, see e.g. [Ste93], page 90). Then

$$\begin{aligned} \frac{1}{R^n} ||\varphi(g - \alpha)||_{H^q}^q &\le \frac{C}{R^n} \int \sup_{t>0} |\phi_t * [\varphi(g - \alpha)](x)|^q \, \mathrm{d}x \\ &= \frac{C}{R^n} \int_{B_{8R}(0)} (\dots) + \frac{C}{R^n} \int_{B_{8R}(0)^c} (\dots) = I + II. \end{aligned}$$

Since $|\phi_t * \varphi(x)| \le 1$, we have

$$|(\phi_t * [\varphi(g - \alpha)])(x)|^q \le |\phi_t * (\varphi g)(x)|^q + |\alpha|^q |\phi_t * \varphi(x)|^q \le C (Mg)^q(x).$$

So for I, we get the estimate $I \le \frac{C}{R^n} \int_{B_{C_0 R}(0)} (Mg)^q(x) \le M([Mg]^q)(z)$ for any $z \in B_{10R}(0)$.

To deal with II, we use the fact that $\int \varphi(g-\alpha) = 0$. Let $B_{8R}(0)^c = \cup_{j\geq 3} R_j$ where $R_j = \{x \in \mathbb{R}^n : |x| \approx 2^j R\}$ and split $II = \sum_j II_j$ accordingly. For $x \in R_j$ and $\phi_t * [\varphi(g-\alpha)](x) \neq 0$, we need $t > \frac{2^j R}{4}$, thus

$$\begin{aligned} II_j &= \frac{1}{R^n} \int_{R_j} \sup_{t>0} |[\phi_t * \varphi(g-\alpha)](x)|^q \, \mathrm{d}x \\ &= \frac{1}{R^n} \int_{R_j} \sup_{t>2^{j-2}R} |\phi_t * [\varphi(g-\alpha)](x)|^q \, \mathrm{d}x \\ &= \frac{1}{R^n} \int_{R_j} \sup_{t>2^{j-2}R} \left| \frac{1}{t^n} \int \left[\phi(\frac{x-y}{t}) - \phi(\frac{x}{t}) \right] [\varphi(g-\alpha)](y) \, \mathrm{d}y \right|^q \, \mathrm{d}x \end{aligned}$$

Since ϕ is smooth and the constant "C" is allowed to depend on bounds for derivatives of ϕ, we get $|\phi(\frac{x-y}{t}) - \phi(\frac{x}{t})| \leq C\frac{|y|}{t}$. Hence

$$II_j \leq \frac{C}{R^n} \int_{R_j} \frac{R}{(2^j R)^{(n+1)q}} \left(\underbrace{\left[\int |\varphi g| \, \mathrm{d}y \right]^q}_{\leq CR^{nq}(Mg)^q(z)} + \underbrace{|\alpha|^q}_{\leq (Mg)^q(z)} \underbrace{\left[\int |\varphi| \, \mathrm{d}y \right]^q}_{\leq CR^{nq}} \right),$$

and therefore,

$$II_j \leq \frac{C}{R^n} (2^j R)^n \frac{R}{(2^j R)^{(n+1)q}} R^{nq} (Mg)^q(z) \leq C 2^{j(n-(n+1)q)} (Mg)^q(z)$$

for any $z \in B_{C_0 R}(0)$. Since $n-(n+1)q < 0$ for $\frac{n}{n+1} < q \leq 1$, we can sum in j, and so $II \leq CM([Mg]^q)(z)$, which completes the proof. □

Theorem 6.3.23. *Assume that $n = 2$ and that $(D)_{p'}^{div \frac{A^T}{detA} \nabla}$ and $(N)_{H^q}^{divA\nabla}$ hold for some $1 < p < \infty$ and $\frac{n}{n+1} < q \leq 1$. Then, $(N)_p^{divA\nabla}$ holds.*

Proof. The proof follows along the same lines as the proof of Theorem 6.3.14. Thus, it is enough to prove the equivalent of Lemma 6.3.15, namely that

$$|E(\tau\lambda)| \leq \varepsilon^{1+\eta} |E_\alpha(\lambda)| + |\{M([Mf]^q) > \gamma\lambda\}|.$$

In order to prove this, one can use the same arguments until the introduction of v. Choose φ as a cut-off for Q_k with $\varphi \equiv 1$ on $6Q_k$. Let u_{new} be the weak solution with Neumann boundary data

$$\Psi = (1-\varphi) f - \frac{1-\varphi}{\chi(1-\varphi)} \left(\int_{\partial B} (1-\varphi) f \right),$$

where $\chi(1-\varphi) = \int_{\partial B} (1-\varphi)$, and let v_{new} be the weak solution with Neumann boundary data for $\Phi = f - \Psi$. Then, $u = u_{new} + v_{new}$. Thus, as in the proof for the Regularity problem, one is left with:

$$|Q_k \cap E(\tau\lambda)| \leq \frac{C}{(\tau\lambda)^{\frac{\bar{p}}{1}}} \int_{5Q_k} N(\nabla u_{new})^{\bar{p}} + \frac{C}{\tau\lambda} \int_{5Q_k} N(\nabla v_{new})^q$$

For the first term, observe that the co-normal derivative of u_{new} is 0 on $6Q_k$. Thus by considering the conjugate $\tilde{u}_{new}$ for u_{new}, we see that the tangential derivative of $\tilde{u}_{new}$ vanishes on $6Q_k$. Since $\tilde{u}_{new}$ is defined up to constants, we can assume that $\tilde{u}_{new} = 0$ on $6Q_k$. So the reverse Hölder inequality, Lemma 6.3.9, is applicable for $\tilde{u}_{new}$, and since $N(\nabla \tilde{u}_{new}) \approx N(\nabla u_{new})$, we have

$$\frac{C}{(\tau\lambda)^{\bar{p}}} \int_{5Q_k} N(\nabla u_{new})^{\bar{p}} \, \mathrm{d}\sigma \leq \frac{C|Q_k|}{(\tau\lambda)^{\bar{p}}} \left(\fint_{6Q_k} N_\alpha(\nabla u_{new}) \right)^{\bar{p}}$$

as in the case of the regularity problem.

For the second term, we use the $(N)_{H^q}$ condition to get

$$\frac{C}{\tau\lambda}\int_{5Q_k} N(\nabla v_{new})^q \leq \frac{C}{\tau\lambda}||\Phi||_{H^q}.$$

For α, as in Lemma 6.3.22, we write

$$\begin{aligned}
\Phi &= \varphi(f-\alpha) + \frac{1-\varphi}{\chi(1-\varphi)}\left[\int_{\partial B}(1-\varphi)f\right] + \varphi\alpha \\
&= \varphi(f-\alpha) + \left(-\frac{1-\varphi}{\chi(1-\varphi)}\left[\int_{\partial B}\varphi f\right] + \frac{\varphi}{\chi(\varphi)}\left[\int_{\partial B}\varphi f\right]\right) = \Phi_I + \Phi_{II}.
\end{aligned}$$

For Φ_I, we apply Lemma 6.3.22 to get

$$||\Phi_I||_{H^q} \leq C\lambda\gamma|Q_k|.$$

Since $||\Phi_{II}||_{L^\infty(\partial B)} \leq C\frac{1}{\chi(\varphi)}|\int_{\partial B}\varphi f|$ and $\int_{\partial B}\Phi_{II} = 0$, we can see $\Phi_{II}/||\Phi_{II}||_\infty$ as an atom corresponding to ∂B and therefore $||\Phi_{II}||_{H^q} \leq \frac{1}{\chi(\varphi)}|\int_{\partial B}\varphi f| \leq C\lambda\gamma|Q_k|$. Thus

$$\frac{C}{\tau\lambda}\int_{5Q_k} N(\nabla v_{new})^q \leq \frac{C}{\tau}\gamma|Q_k|$$

and so, one can proceed to finish the proof as in the $(R)_{\mathcal{C}^q}$ case. $\square$

Chapter 7

Open Problems

In this last chapter, we will summarize some open problems we have seen throughout the thesis.

- In Chapter 2, we studied the continuous Dirichlet problem for elliptic operators with drift terms that satisfy the growth condition $\frac{\varepsilon_1}{\delta(X)}$ for ε_1 small. It remains an open problem if the smallness of ε_1 is generally essential. The improvement (compared to [GT01]) to allow unbounded drift terms uses the estimate

$$\int_\Omega |u(X)||B(X)||\nabla u(X)| \,\mathrm{d}X \leq (\varepsilon + C_\varepsilon \varepsilon_1) \int_\Omega |\nabla u(X)|^2 \,\mathrm{d}X + C \int_\Omega u(X)^2 \,\mathrm{d}X \tag{7.1}$$

  (see the proof of Lemma 2.2.4) for $u \in W_0^{1,2}(\Omega)$ and then the smallness of ε and ε_1 to hide the first term in a term of the form $\int_\Omega |\nabla u|^2$ gained by the ellipticity assumption. By the following example in one dimension, we see that the smallness of ε_1 is essential to this hiding step: Let $\Omega = [0, 1]$,

$$f(x) = \begin{cases} x & \text{on } [0, \delta] \\ \delta & \text{on } [\delta, 1-\delta] \\ 1 - x & \text{on } [1-\delta, 1] \end{cases}$$

  and choose $B(x) = \frac{\mathrm{sgn}(f'(x))}{x} \varepsilon_1$ for $x \in [0, \frac{1}{2}]$ with a symmetric extension around $\frac{1}{2}$, then

$$\int_0^1 B(x) f(x) f'(x) \,\mathrm{d}x = 2\varepsilon_1 \delta, \qquad \int_0^1 (f'(x))^2 \,\mathrm{d}x = 2\delta, \qquad \int_0^1 f(x)^2 \,\mathrm{d}x \approx \delta^2$$

  and therefore the estimate (7.1) reads as

$$2\varepsilon_1 \delta \leq (\varepsilon + C_\varepsilon \varepsilon_1) 2\delta + C\delta^2.$$

  By choosing δ sufficiently small, we see that the smallness of ε_1 for the mentioned hiding step is essential. On the other hand, one can use Theorem 1.9 in Chapter III of [HL01] to demonstrate the existence of an elliptic operator of the form $L = \mathrm{div} A\nabla - B\nabla$ such that $\delta(X)|B(X)|^2 \,\mathrm{d}X$ is a Carleson measure and the drift term B satisfies only $\frac{C}{\delta(X)}$ for a non-small C in $\Gamma(Q), Q \in \partial\Omega$ and the corresponding continuous Dirichlet problem is solvable.

- Motivated by the fact that $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} < \infty$ implies boundary values in $W^{1,p}(\partial\Omega)$ (see Theorem 6.2.1), it remains an open question whether the boundary value f guaranteed by $||N(\nabla u)||_{L^1(\partial\Omega)} + ||u||_{L^1(\Omega)} < \infty$ (see Theorem 6.2.2) is in HS^1.

- Does $(N)_{H^1}$ and $(R)_{\mathrm{HS}^1}$ (or $(D^*)_{\mathrm{BMO}}$) imply $(N)_p$ for some $1 < p < \infty$?

- We were only able to include the endpoint and the range $0 < q < 1$ for Z. Shen's main Theorem in [She07]. Thus, it is still an open problem whether $(D^*)_{p'}$ implies $(R)_p$.

On the one hand, Corollary 6.3.13 simplifies the requirements for a possible proof a lot. On the other hand, the $(D)_p$ condition is used to show that there is a unique function u with $||u^*||_{L^p(\partial\Omega)} < \infty$ and u converges to zero non-tangentially almost everywhere, whereas for the proof of the uniqueness in the $(R)_p$ case, only the $||N(\nabla u)||_{L^p(\partial\Omega)} + ||u||_{L^p(\Omega)} < \infty$ condition is needed to conclude uniqueness, not the $(R)_p$ condition.

Appendix A

Appendix

For completeness, we include here a summary of results, which can be found in several monographs and which were used in the thesis.

Theorem A.0.24. *[Fredholm Alternative, Theorem 5.3 in [GT01]] Let T be a compact linear mapping of a normed linear space X into itself. Then either the homogeneous equation $x - Tx = 0$ has a non-trivial solution $x \in X$ or for each $y \in X$ the equation $x - Tx = y$ has a unique solution $x \in X$. In the second case, the inverse of the operator $(id - T)$ exists and is bounded.*

Lemma A.0.25. *Let A be a bounded measurable set in $\mathbb{R}^n$ with $|A| \neq 0$ and u be a measurable function with $|u|^p \in L^1(A)$ for all $p \in \mathbb{R}$. We define $\Phi(u,p) = \left(\fint_A |u|^p\right)^{\frac{1}{p}}$, then*

$$\lim_{p\to\infty} \Phi(u,p) = \sup_A |u|$$
$$\lim_{p\to-\infty} \Phi(u,p) = \inf_A |u|.$$

Proof. We know that $\lim c^{\frac{1}{n}} \to 1$ for any $c > 0$. Hence, for any $\delta > 0$ there exists $p = p(\delta, \varepsilon)$ such that $|1 - \delta^{\frac{1}{p}}| \leq \varepsilon$. Let $M = \sup_A |u|$. If $M = \infty$, then there exist measurable sets $E_k \subset A$ and constants C_k such that $|E_k| \geq C_k$ and $|u| \geq k$ on E_k. Thus for $p_k = p_k(C_k)$ large, we get

$$\left(\fint_A |u|^{p_k}\right)^{\frac{1}{p_k}} \geq k \left(\frac{C_k}{|A|}\right)^{\frac{1}{p_k}} \geq k(1 - \frac{1}{k}).$$

By sending $k \to \infty$, we get $\lim_{p\to\infty} \Phi(u,p) = \infty$. For $M < \infty$, there exist measurable sets E_k and constants C_k as before, such that $|u| \geq M - \frac{1}{k}$ on E_k. Hence

$$\left(\fint_A |u|^{p_k}\right)^{\frac{1}{p_k}} \geq (M - \frac{1}{k}) \left(\frac{C_k}{|A|}\right)^{\frac{1}{p_k}} \geq k(1 - \frac{1}{k}).$$

Sending $k \to \infty$ finishes the proof regarding the supremum result.
For the second part of the Lemma, we assume first that $\inf |u| > 0$. Then, for $p < 0$, we have

$$\Phi(u,p) = \frac{1}{\left(\fint_A (\frac{1}{|u|})^{|p|}\right)^{\frac{1}{|p|}}} \to \frac{1}{\sup_A \frac{1}{|u|}} = \inf_A |u|.$$

If $\inf_A |u| = 0$, we consider $|u| + \varepsilon$ and send ε to zero. □

Lemma A.0.26 (Lemma 8.23 in [GT01])**.** *Let w be a non-decreasing function on an interval $(0, R_0]$ satisfying the inequality*

$$w(\tau R) \leq \gamma w(R) + \sigma(R)$$

for all $R \leq R_0$, where σ is non-decreasing and $0 < \gamma, \tau < 1$. Then, for any $\mu \in (0,1)$ and

$R \le R_0$, we have

$$w(R) \le C\left(\left(\frac{R}{R_0}\right)^{\alpha} w(R_0) + \sigma(R^{\mu} R_0^{1-\mu})\right),$$

where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants.

Theorem A.0.27 (Theorem 7.21 in [GT01])**.** *Let $u \in W^{1,1}(\Omega)$ where Ω is convex, and suppose there exists a constant K such that*

$$\int_{\Omega \cap B_R} |Du|\, dx \le K R^{n-1}$$

for all balls B_R. Then, there exist positive constants σ_0 and C depending only on n such that

$$\int_{\Omega} \exp\left(\frac{\sigma}{K}|u - u_{\Omega}|\right)\, dx \le C(diam\Omega)^n,$$

where $\sigma = \sigma_0|\Omega|(diam\Omega)^{-n}$.

Lemma A.0.28. *If $T : B_1 \to B_2$ is a surjective bounded linear operator from the Banach space B_1 to the Banach space B_2, the set $T(S)$ is dense in B_2 for any dense subset $S \subset B_1$ of B_1 .*

Proof. Assume that $x \in \overline{TS}^c \subset B_2$. Then there exists an open set $A \subset B_2$ such that $x \in A$ and $A \cap \overline{TS} = \emptyset$. Since T is continuous and surjective, we have an open and non-empty $T^{-1}A$, which contradicts the density of S. □

Theorem A.0.29 (John–Nirenberg's inequality, see [Gra09], page 128)**.** *Let $f \in BMO$ and $\gamma < \frac{1}{2^n e}$. Then, we have*

$$⨍_Q \exp\left(\frac{\gamma|f - f_Q|}{||f||_{BMO}}\right) \le 1 + \frac{2^n e^2 \gamma}{1 - e^n e\gamma}$$

for all cubes Q.

Theorem A.0.30 (Jensen's inequality, see [Eva98], page 621)**.** *Assume $f : \mathbb{R} \to \mathbb{R}$ is convex, and $U \subset \mathbb{R}^n$ is open and bounded. Let $u : U \to \mathbb{R}$ be integrable. Then*

$$f\left(⨍_U u\, dx\right) \le ⨍_U f(u)\, dx.$$

Bibliography

[Ada75] R.A. Adams. *Sobolev spaces*. Academic Press, New York, 1975.

[BA56] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Mathematica*, 96(1):125–142, 1956.

[Bau84] P. Bauman. Positive solutions of elliptic equations in nondivergence form and their adjoints. *Arkiv för Matematik*, 22(1):153–173, 1984.

[BB10] N. Badr and F. Bernicot. Abstract hardy-sobolev spaces and interpolation. *Journal of Functional Analysis*, 259(5):1169–1208, 2010.

[BD09] N. Badr and G. Dafni. An atomic decomposition of the Hajłasz Sobolev space M_1^1 on manifolds. *Arxiv preprint arXiv:0910.2895*, 2009.

[BMR99] J. Bastero, M. Milman, and FJ Ruiz. Reverse Hölder inequalities and interpolation. *Israel Math. Conf. Proc*, 13:11–23, 1999.

[Buc93] S.M. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Transactions of the American Mathematical Society*, 340(1):253–272, 1993.

[Cal72] A.P. Calderón. Estimates for singular integral operators in terms of maximal functions. *Studia Math*, 44(6):167–186, 1972.

[CFK81] L. Caffarelli, E. Fabes, and C.E. Kenig. Completely singular elliptic-harmonic measures. *Indiana Univ. Math. J*, 30:917–924, 1981.

[CFMS81] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa. Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ. Math. J*, 30(4):621–640, 1981.

[CR80] R.R. Coifman and R. Rochberg. Another characterization of BMO. *Proceedings of the American Mathematical Society*, 79(2):249–254, 1980.

[Dah77] B.E.J. Dahlberg. Estimates of harmonic measure. *Archive for Rational Mechanics and Analysis*, 65(3):275–288, 1977.

[Dah79] B.E.J. Dahlberg. On the Poisson integral for Lipschitz and C^1-domains. *Studia Math.*, 66:13–24, 1979.

[Dah86] B.E.J. Dahlberg. On the absolute continuity of elliptic measures. *American Journal of Mathematics*, pages 1119–1138, 1986.

[DCU01] J. Duoandikoetxea and D. Cruz-Uribe. *Fourier analysis*. American Mathematical Society, 2001.

[DJK84] B.E.J. Dahlberg, D.S. Jerison, and C.E. Kenig. Area integral estimates for elliptic differential operators with non-smooth coefficients. *Arkiv för Matematik*, 22(1):97–108, 1984.

[DKP10] M. Dindos, C.E. Kenig, and J. Pipher. BMO solvability and the A_∞ condition for elliptic operators. *Arxiv preprint arXiv:1007.5496*, 30, 2010.

[Doo94] J.L. Doob. *Measure theory.* Springer, 1994.

[DPP07] M. Dindos, S. Petermichl, and J. Pipher. The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function. *Journal of Functional Analysis*, 249(2):372–392, 2007.

[DS84] R.A. DeVore and R.C. Sharpley. *Maximal functions measuring smoothness.* 293. Amer Mathematical Society, 1984.

[Eva98] L.C. Evans. Partial differential equations. *American Mathematical Society, Providence, RI*, 1998.

[Fef89] R. Fefferman. A criterion for the absolute continuity of the harmonic measure associated with an elliptic operator. *Journal of the American Mathematical Society*, 2(1):127–135, 1989.

[FJK84] E.B. Fabes, D.S. Jerison, and C.E. Kenig. Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure. *Annals of Mathematics*, 119(1):121–141, 1984.

[FJR78] EB Fabes, M. Jodeit, and NM Riviere. Potential techniques for boundary value problems on C^1-domains. *Acta Mathematica*, 141(1):165–186, 1978.

[FKP91] R.A. Fefferman, C.E. Kenig, and J. Pipher. The theory of weights and the dirichlet problem for elliptic equations. *Annals of Mathematics*, 134(1):65–124, 1991.

[Fol95] G.B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, NJ, second edition, 1995.

[FS72] C. Fefferman and E.M. Stein. H^p spaces of several variables. *Acta Mathematica*, 129(1):137–193, 1972.

[Geh73] F.W. Gehring. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Mathematica*, 130(1):265–277, 1973.

[Gia83] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Princeton Univ Pr, 1983.

[Gra08] L. Grafakos. *Classical Fourier Analysis.* Springer Verlag, 2008.

[Gra09] L. Grafakos. *Modern Fourier Analysis.* Springer Verlag, 2009.

[GT01] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer Verlag, 2001.

[GW82] M. Grüter and K.O. Widman. The Green function for uniformly elliptic equations. *Manuscripta Mathematica*, 37(3):303–342, 1982.

[Haj03] P. Hajłasz. Sobolev spaces on metric-measure spaces. *Amer Mathematical Society*, 338:173–218, 2003.

[Har20] G.H. Hardy. Note on a theorem of hilbert. *Mathematische Zeitschrift*, 6(3):314–317, 1920.

[Hel75] L.L. Helms. Introduction to potential theory. *Pure and Applied Mathematics*, 22, 1975.

[HL01] S. Hofmann and J.L. Lewis. The Dirichlet problem for parabolic operators with singular drift terms. *Amer Mathematical Society*, 151(719), 2001.

[IR05] A. Ifra and L. Riahi. Estimates of green functions and harmonic measures for elliptic operators with singular drift terms. *Publ. Math*, 49:159–177, 2005.

[JJ94] P.W. Jones and J.L. Journe. On weak convergence in $H^1(R^d)$. *Proceedings of the American Mathematical Society*, 120(1):137–138, 1994.

[Jou83] J.L. Journé. Calderón–zygmund operators, pseudo-differential operators, and the cauchy integral of calderón. *Lect. Notes Math.*, 994, 1983.

[Ken94] C.E. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*, volume 83. Amer Mathematical Society, 1994.

[KKPT00] C.E. Kenig, H. Koch, J. Pipher, and T. Toro. A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Advances in Mathematics*, 153(2):231–298, 2000.

[KP93] C.E. Kenig and J. Pipher. The Neumann problem for elliptic equations with non-smooth coefficients. *Inventiones Mathematicae*, 113(1):447–509, 1993.

[KP01] C.E. Kenig and J. Pipher. The Dirichlet problem for elliptic equations with drift terms. *Publ. Mat*, 45(1):199–217, 2001.

[KR09] C.E. Kenig and D.J. Rule. The regularity and Neumann problem for non-symmetric elliptic operators. *American Mathematical Society*, 361(1):125–160, 2009.

[LSW63] W. Littman, G. Stampacchia, and H.F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 17(1-2):43–77, 1963.

[Mil96] M. Milman. A note on Gehring's lemma. *Acad. Sci. Fenn. Math.*, 21:389–398, 1996.

[Miy90] A. Miyachi. Hardy-Sobolev spaces and maximal functions. *Journal of the Mathematical Society of Japan*, 42(1):73–90, 1990.

[MT99] M. Mitrea and M. Taylor. Boundary Layer Methods for Lipschitz Domains in Riemannian Manifolds. *Journal of Functional Analysis*, 163(2):181–251, 1999.

[MT00] M. Mitrea and M. Taylor. Potential theory on Lipschitz domains in Riemannian manifolds: Hölder continuous metric tensors. *Communications in Partial Differential Equations*, 25(7-8):1487–1536, 2000.

[MT01] M. Mitrea and M. Taylor. Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy, and Hölder space results. *Communications in Analysis and Geometry*, 9(2):369–422, 2001.

[Muc72] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Transactions of the American Mathematical Society*, 165:207–226, 1972.

[Ros62] M. Rosenblum. Summability of Fourier Series in $L^p(d\mu)$. *Transactions of the American Mathematical Society*, 105(1):32–42, 1962.

[Rud66] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1966.

[She07] Z. Shen. A relationship between the Dirichlet and regularity problems for elliptic equations. *Math. Res. Lett.*, 14(2):205–213, 2007.

[Sta65] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier*, 15(1):189–258, 1965.

[Ste69] E.M. Stein. Note on the class L log L. *Studia Math*, 32:305–310, 1969.

[Ste70] E.M. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, N.J., 1970.

[Ste93] E.M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43. Princeton University Press, Princeton, NJ, 1993.